Capacitive Sensors

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1 Introduction

Many MEMS devices contains capacitive elements as part of either actuation or sensing schemes. Capacitive elements are used in pressure sensors, accelerometers, moving mirrors and many other devices. This short lecture note describes capacitive elements based on the well known parallel plate capacitor.

This lecture note is brand new so please report any errors or good ideas for improvements. Some of the figures are made by Bernard Legrand as part of his lecture "Electrostatic actuation".

2 The parallel plate capacitor

This section summarizes well known results for parallel plate capacitors where the dimensions of the plates are much larger than the plate distance such that fringing fields can be ignored. The capacitance of two plates having area S separated a distance d is

$$C = \frac{\epsilon S}{d} \tag{1}$$

where ϵ is the dielectric constant of the material between the plates (often vacuum where $\epsilon = \epsilon_0$). The charge, Q, and the applied voltage, V, are related by

$$Q = CV \tag{2}$$

The electric field, E, between the plates is

$$E = \frac{V}{d} = \frac{Q}{Cd} = \frac{Q}{\epsilon S} \tag{3}$$

and the stored energy in the capacitor, U_e, is

$$U_e = \frac{1}{2}CV^2 \tag{4}$$

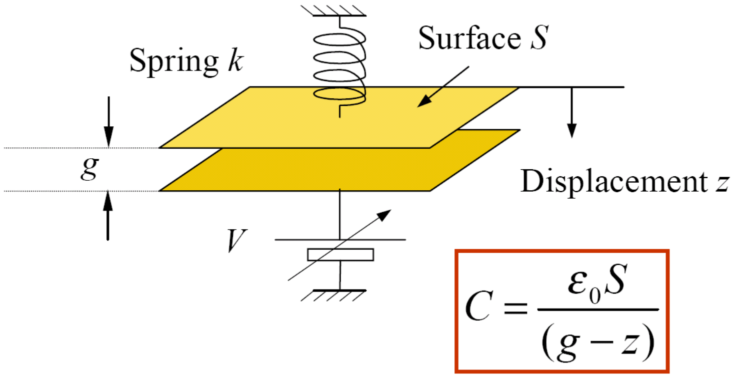


Figure 1: A capacitor with a moving plate. The initial gap is g and the spring has spring constant k.

The force between the plates is calculated from the potential energy as

$$F = -\frac{\partial U_e}{\partial d} = -\frac{\partial}{\partial d}\left(\frac{1}{2}CV^2\right) = -\frac{\partial}{\partial d}\left(\frac{1}{2}\frac{\epsilon S}{d}V^2\right)$$

yielding

$$F = \frac{1}{2}\frac{\epsilon S}{d^2}V^2 \tag{5}$$

3 Capacitive sensing

Fig. 1 shows a parallel plate capacitor where one of the plates are attached to a spring so it can move in the z direction. This spring could be part of a mechanical element in a MEMS device such as a plate. The capacitance is given by

$$C(z) = \frac{\epsilon S}{g - z} \tag{6}$$

which for $z = 0$ gives the zero point capacitance, C_0, as

$$C_0 = \frac{\epsilon S}{g} \tag{7}$$

The ratio of the capacitance to zero point capacitance is

$$\frac{C}{C_0} = \frac{\frac{\epsilon S}{g-z}}{\frac{\epsilon S}{g}} = \frac{g}{g-z} = \frac{1}{1-\frac{z}{g}} = \frac{1}{1-u} \tag{8}$$

where the relative displacement u is given by

$$u = \frac{z}{g} \tag{9}$$

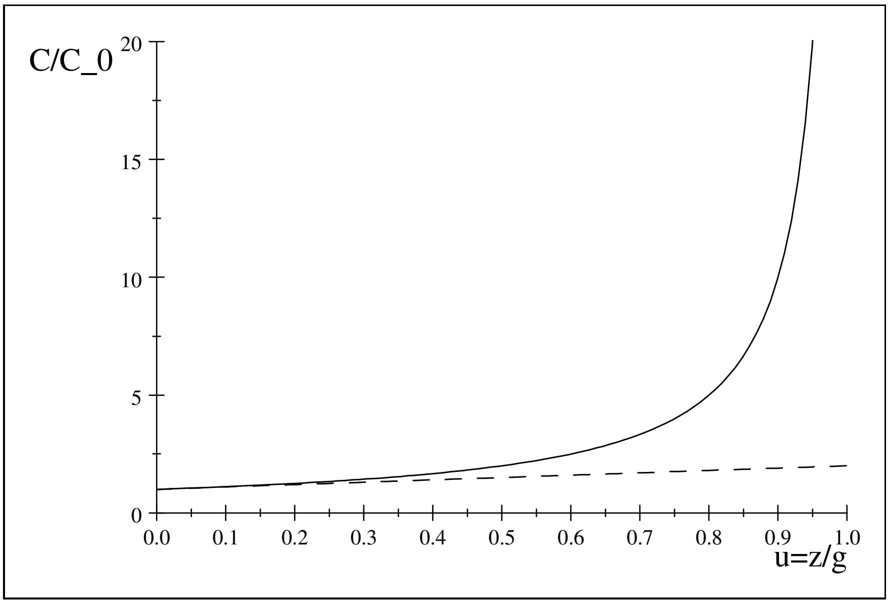


Figure 2: Plot of $\frac{C}{C_0}$ as function of $u = z/g$ (Eq. 8, solid line) and the Taylor expansion (Eq. 10, dashed line).

This expression is plotted on Fig. 2. For small values of u we can make a Taylor expansion and obtain

$$\frac{C}{C_0} = \frac{1}{1-u} \approx 1 + u \tag{10}$$

which is plotted as the dashed line on Fig. 2.

The change in capacitance, ΔC, is defined as

$$\Delta C = C - C_0 \tag{11}$$

and the relative change in capacitance is

$$\frac{\Delta C}{C_0} = \frac{C - C_0}{C_0} = \frac{C}{C_0} - 1 = \frac{1}{1 - \frac{z}{g}} - 1 = \frac{z}{g - z} = \frac{\frac{z}{g}}{1 - \frac{z}{g}} = \frac{u}{1-u} \tag{12}$$

which is plotted on Fig. 3. Clearly, this type of sensor is not linear. For small values of z we can make a Taylor expansion and obtain

$$\begin{aligned} \frac{\Delta C}{C_0} &= \frac{z}{g-z} = \frac{u}{1-u} = \frac{1}{g}z + \frac{1}{g^2}z^2 + O\left(z^3\right) \\ &\approx u + u^2 \end{aligned} \tag{13}$$

which is plotted as the dashed lines on Figs. 3 and 4. For small values of u a linear response is obtained.

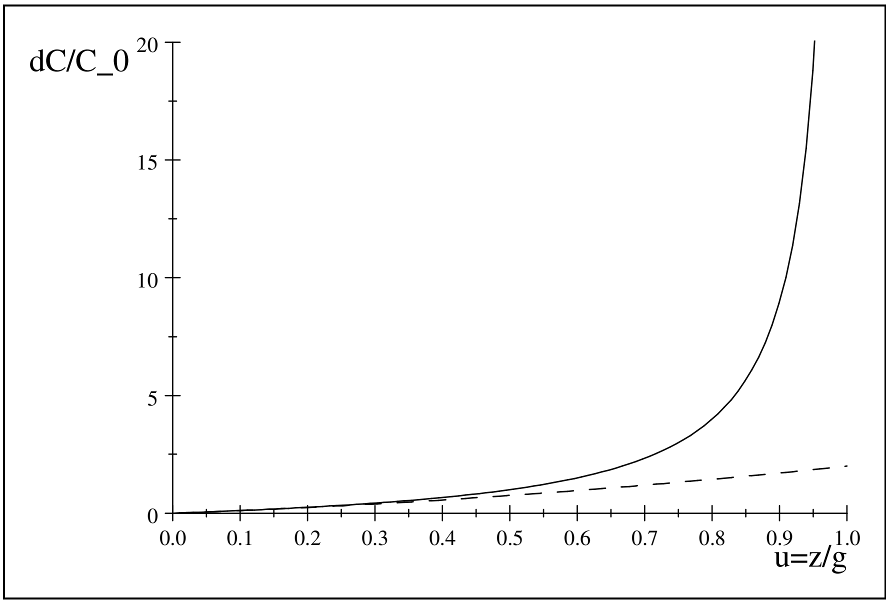


Figure 3: Plot of $\frac{\Delta C}{C_0}$ as function of $u = z/g$ (Eq. 12, solid line) and the Taylor expansion (Eq. 13 dashed line).

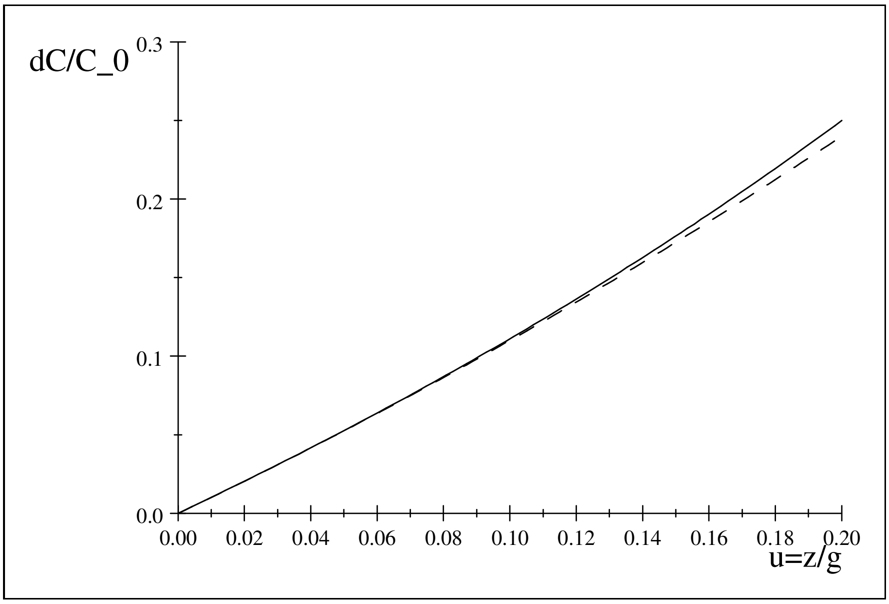


Figure 4: Plot of $\frac{\Delta C}{C_0}$ as function of $u = z/g$ (solid line) and the Taylor expansion (dashed line) for small values of u. Note the linear region.

4 Pull in voltage

When a capacitive sensor is used a voltage is applied across the plates of the capacitor. This gives an *attractive* force between the plates. This force will pull the two plates together and elongate the spring. To find the equilibrium condition we will use an energy method to find the total force on the moving plate. Equilibrium is obtained when the total force on the moving plate is zero.

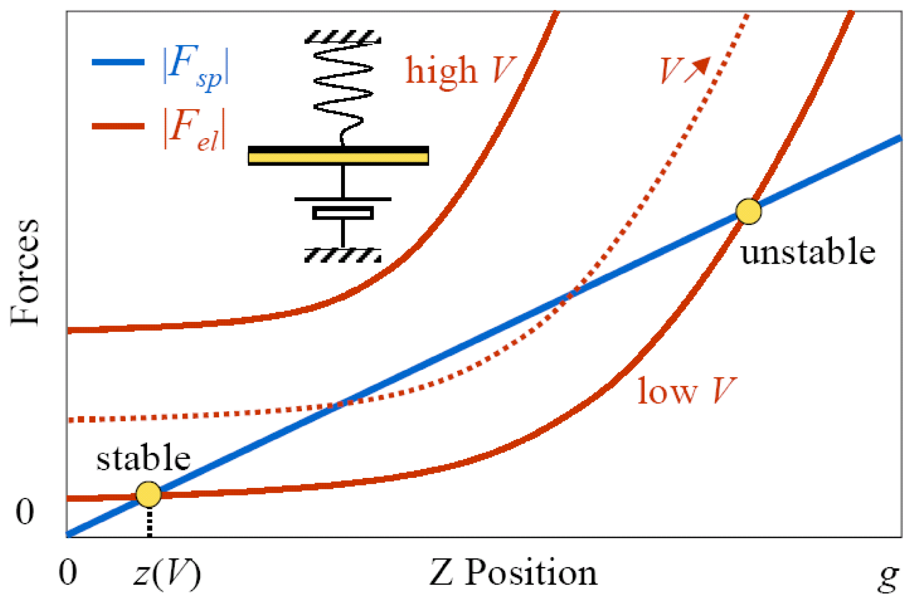


Figure 5: Plot of the spring force, F_{sp}, and the electrostatic force, F_{el}, between the plates. At high voltages the electrostatic force is always larger the the spring force and the moving plate is pulled down to the fixed plate. At lower voltages there is a stable position where the electrostatic force equals the spring force.

The potential energy, U, is (using Eq. 6 for $C(z)$)

$$U = -\frac{1}{2}CV^2 + \frac{1}{2}kz^2 = -\frac{1}{2}\frac{\epsilon S}{g-z}V^2 + \frac{1}{2}kz^2 \tag{14}$$

The force is obtained from the potential energy as

$$F = -\frac{\partial U}{\partial z} \tag{15}$$

$$= -\frac{\partial}{\partial z}\left(-\frac{1}{2}\frac{\epsilon S}{g-z}V^2 + \frac{1}{2}kz^2\right)$$

$$= -\frac{\partial}{\partial z}\left(-\frac{1}{2}\frac{\epsilon S}{g-z}V^2\right) - \frac{\partial}{\partial z}\left(+\frac{1}{2}kz^2\right)$$

$$= \frac{1}{2}\frac{\epsilon S}{(g-z)^2}V^2 - kz \tag{16}$$

The first term is the electrostatic force, F_{el}, between the plates which is proportional to V^2 and the second term is the force, F_{sp}, corresponding to the spring which depends linearly on z. These forces are sketched on Fig. 5. At high voltages the electrostatic force is always larger than the spring force and the moving plate is pulled down to the fixed plate. At lower voltages there is a *stable* position where the electrostatic force equals the spring force, i.e. the total force is zero, and an *unstable* position as well.

In order to investigate the stable and unstable positions we will examine the conditions for a stable equilibrium position. If the plate is moved a small distance δz and the increase in the force, δF, is positive then the two plates will be pulled together. We can write the change in force with displacement as

$$\delta F = \left.\frac{\partial F}{\partial z}\right|_V \delta z \tag{17}$$

and examine the derivative by using Eq. 16

$$\begin{aligned}\left.\frac{\partial F}{\partial z}\right|_V &= \frac{\partial}{\partial z}\left(\frac{1}{2}\frac{\epsilon S}{(g-z)^2}V^2 - kz\right) \\ &= \frac{\partial}{\partial z}\left(\frac{1}{2}\frac{\epsilon S}{(g-z)^2}V^2\right) + \frac{\partial}{\partial z}(-kz) \\ &= \frac{\epsilon S}{(g-z)^3}V^2 - k \end{aligned} \tag{18}$$

In order for this to be negative such that a stable position is found we must have

$$\frac{\epsilon S}{(g-z)^3}V^2 - k < 0 \tag{19}$$

from which we obtain

$$\frac{\epsilon S}{(g-z)^3}V^2 < k \tag{20}$$

This puts a requirement on the largest voltage that we can apply to have a stable position. This voltage is called the pull-in voltage, V_{pi}, and is given by

$$\frac{\epsilon S}{(g-z)^3}V_{pi}^2 = k \tag{21}$$

At the equilibrium point the electrostatic and the spring force equals each other

$$\frac{1}{2}\frac{\epsilon S}{(g-z)^2}V_{pi}^2 = kz \tag{22}$$

Combining Eqns. 21 and 22 we obtain

$$\begin{aligned}
\frac{\epsilon S}{(g-z)^3} V_{pi}^2 &= \frac{1}{2}\frac{1}{z}\frac{\epsilon S}{(g-z)^2} V_{pi}^2 \\
\frac{1}{(g-z)^3} &= \frac{1}{2}\frac{1}{z}\frac{1}{(g-z)^2} \\
\frac{2z}{g-z} &= 1
\end{aligned} \tag{23}$$

The solution to this equation is

$$z = \frac{1}{3} g \tag{24}$$

The voltage corresponding to this point is the pull in voltage given by

$$V_{pi} = \sqrt{\frac{2(g-z)^2}{\epsilon S} kz} = \sqrt{\frac{2\left(g-\frac{1}{3}g\right)^2}{\epsilon S} k \frac{1}{3} g} = \sqrt{\frac{8}{27}\frac{kg^3}{\epsilon S}} = \sqrt{\frac{8}{27}\frac{kg^2}{C_0}} \tag{25}$$

Notice that the pull in voltage depends on $g^{3/2}$.

We will now further investigate the pull situation. At equilibrium the total force is zero, $F = 0$, and

$$\frac{1}{2}\frac{\epsilon S}{(g-z)^2} V^2 - kz = 0 \tag{26}$$

Using Eq. 9 we can rewrite Eq. 26 as

$$\begin{aligned}
\frac{1}{2}\frac{\epsilon S}{(g-gu)^2} V^2 - kgu &= 0 \\
\frac{1}{2}\frac{\epsilon S}{g^2(1-u)^2} V^2 - kgu &= 0 \\
\frac{1}{2}\frac{\epsilon S}{kg^3} V^2 &= u(1-u)^2
\end{aligned} \tag{27}$$

and

$$y = \frac{1}{2}\frac{\epsilon S}{kg^3} V^2 = u(1-u)^2 \tag{28}$$

This expression is plotted on Figs. 6 and 7. The maximum of this curve is given by

$$\begin{aligned}
\frac{d}{du}\left(u(1-u)^2\right) &= 0 \\
3u^2 - 4u + 1 &= 0
\end{aligned}$$

which has solution

$$u = \frac{1}{3} \tag{29}$$

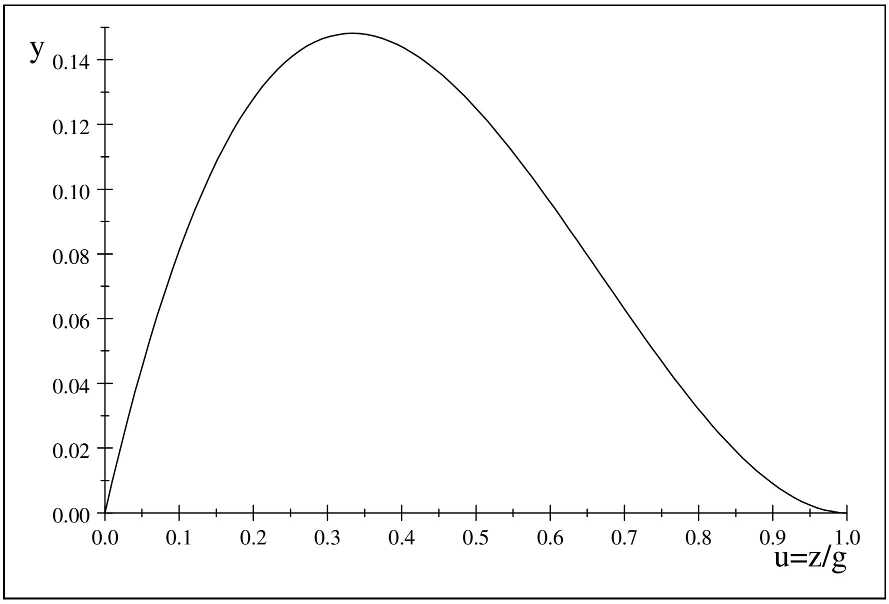


Figure 6: Plot of Eq. 28. The maximum occurs at $u = 1/3$.

At the extremum, where the value of the voltage reaches the pull in voltage, $V = V_{pi}$, the value of y is

$$y = \frac{1}{2}\frac{\epsilon S}{kg^3}V_{pi}^2 = \frac{1}{3}\cdot\left(1 - \frac{1}{3}\right)^2 = \frac{4}{27} = 0.14815 \tag{30}$$

from which we again obtain the pull in voltage as

$$\frac{1}{2}\frac{\epsilon S}{kg^3}V_{pi}^2 = \frac{4}{27} \tag{31}$$

$$V_{pi} = \sqrt{\frac{8}{27}\frac{kg^3}{\epsilon S}} \tag{32}$$

Fig 7 is another representation of the equilibrium conditions. For z values smaller than $g/3$ a stable position can be obtained.

5 Pull out voltage

Once the two plates are pulled together the voltage has to be lowered in order for the plates to separate again. When the mobile plate touches the fixed plate, the distance between the two plates is just the thickness, t_{ox}, of the insulating layer as illustrated on Fig. 8. In this situation the capacitance of the system is

$$C = \frac{\epsilon_{ox} S}{t_{ox}} \tag{33}$$

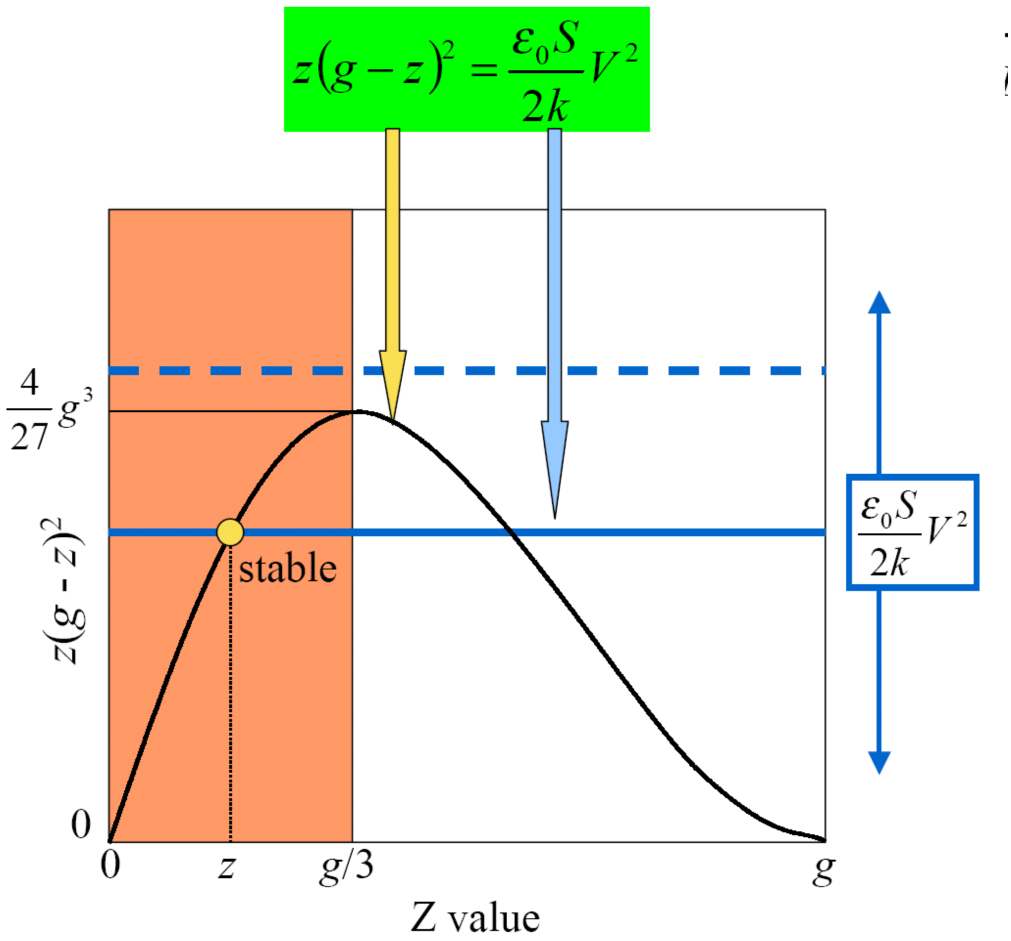


Figure 7: Plot of Eq. 28. The maximum occurs at $z = g/3$ and below this value a stable condition is obtained.

The electrostatic force between the two plates is then

$$F = \frac{1}{2}\frac{\epsilon_{ox}S}{t_{ox}^2}V^2 \tag{34}$$

In order for the two plates to separate again the voltage has to be lowered such that the force from the spring, $F_s = kg$, is larger than the electrostatic force. The voltage where this occurs is the pull out voltage, V_{po} given by

$$\frac{1}{2}\frac{\epsilon_{ox}S}{t_{ox}^2}V_{po}^2 = kg \tag{35}$$

from which we obtain,

$$V_{po} = \sqrt{\frac{2kg}{\epsilon_{ox}S}}t_{ox} \tag{36}$$

Notice, that the pull out voltage is proportional to the thickness of the insulating layer.

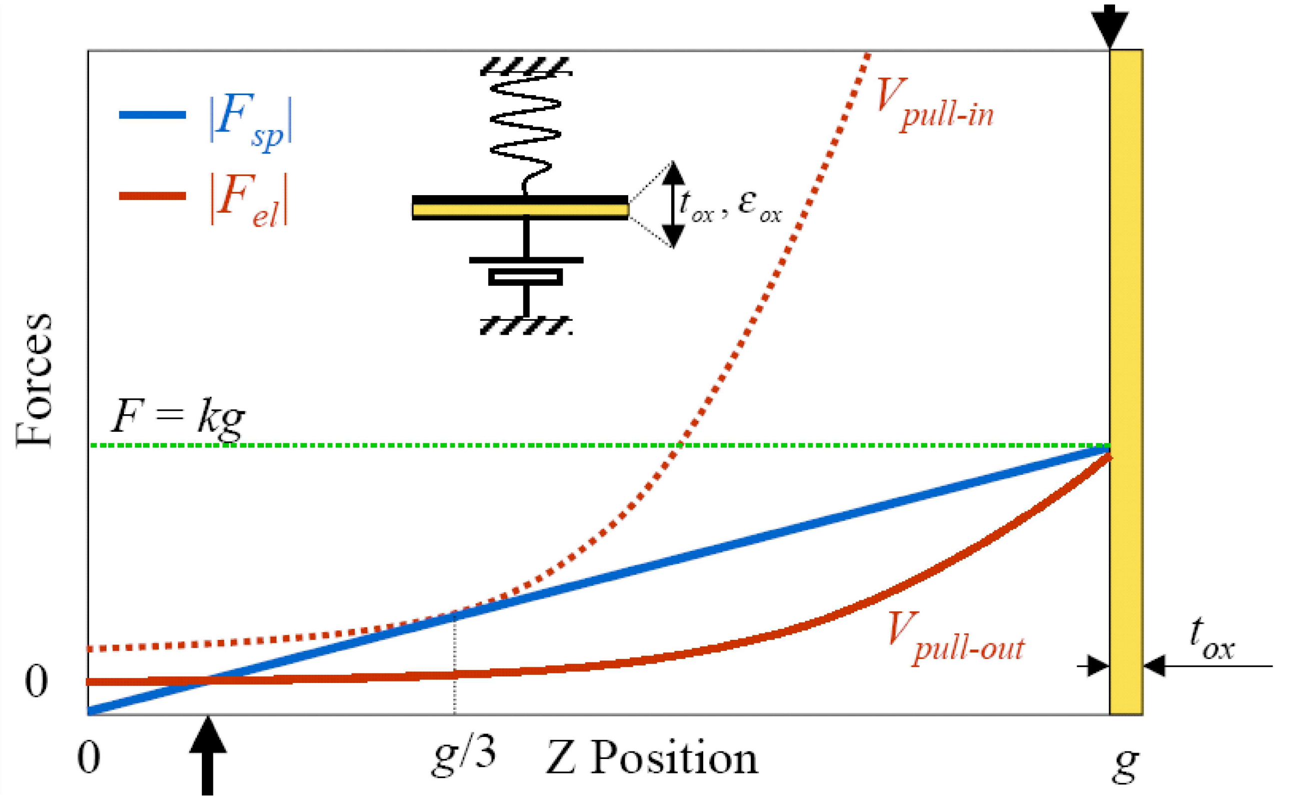


Figure 8: At pull in the moving plate touches the fixed pate and the distance between the two pates is the thickness of the oxide used as an insulation layer.

6 Hysteresis

We can now identify four regions as illustrated on Fig. 9:

Region 1 Where $V < V_{pi}$ and as the voltage is increased the gap is reduced until the pull in voltage is reached

Region 2 For $V > V_{pi}$ the moving plate is pulled down until it reaches the insulating oxide

Region 3 When the voltage is reduced the moving plate stays in the pulled down position for $V_{po} < V < V_{pi}$

Region 4 Finally, when the voltage is lower than the pull out voltage, $V < V_{po}$, it returns to the original position.

Thus, this system has a hysteresis behavior.

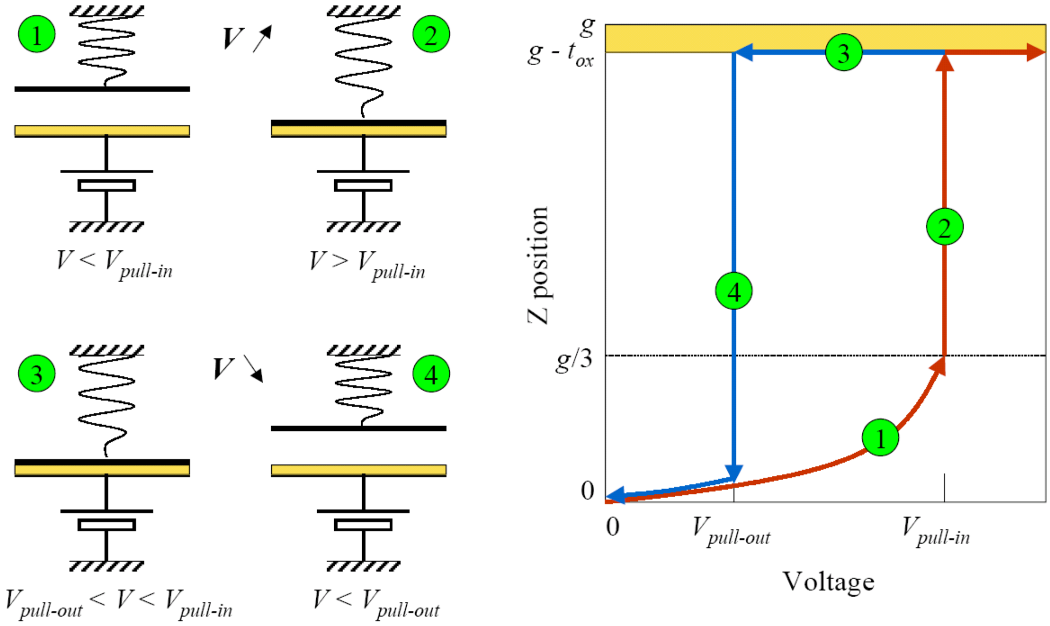


Figure 9: The system shows a hysteresis behavior depending on the voltages applied.

7 CV curve

The capacitance-voltage curve for a capacitive device is easily measured experimentally and it is therefore interesting to find an analytical expression that describes how the capacitance depends on the applied voltage.

The total force is

$$F = \frac{1}{2}\frac{\epsilon S}{(g-z)^2}V^2 - kz$$

and a stable position has

$$\frac{1}{2}\frac{\epsilon S}{(g-z)^2}V^2 - kz = 0.$$

For small deflections where $z \ll g$ we have that $g - z \approx g$ so the force balance becomes

$$\frac{1}{2}\frac{\epsilon S}{g^2}V^2 - kz = 0$$

and solving for z we find

$$z = \frac{1}{2}\frac{\epsilon S}{kg^2}V^2 = \frac{1}{2}\frac{C_0}{kg}V^2.$$

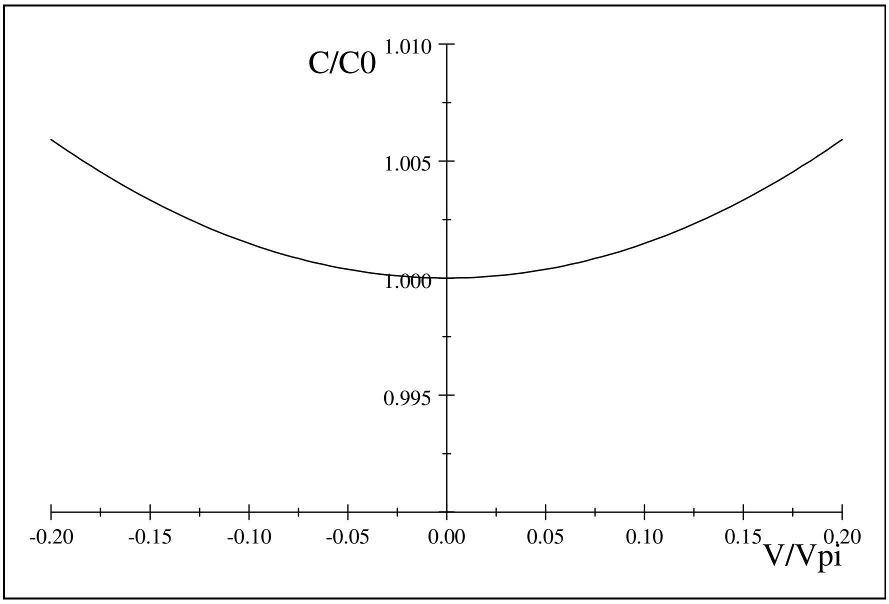
Using that $V_{pi}^2 = \frac{8}{27}\frac{kg^2}{C_0}$ we have $kg = \frac{27}{8}C_0 V_{pi}^2/g$ such that

$$z = \frac{1}{2}\frac{C_0}{kg}V^2 = \frac{1}{2}\frac{C_0}{\frac{27}{8}C_0 V_{pi}^2/g}V^2 = \frac{4}{27}g\frac{V^2}{V_{pi}^2}$$

The capacitance then becomes

$$C = \frac{C_0}{1 - z/g} \approx C_0\left(1 + \frac{z}{g}\right) = C_0\left(1 + \frac{4}{27}\frac{V^2}{V_{pi}^2}\right). \tag{37}$$

This expression shows that the CV curve is a parabula with a curvature of $\frac{4}{27}/V_{pi}^2$. The expression is, however, only valid for small z/g but this is often true practice. Fig. 7 shows a plot of Eq. 37.



The CV curve is a parabula.

8 Problems

Problem 1 Consider a capacitor with plate area of 100×100 μm having a gap distance of 2 μm. The capacitor has vacuum between the plates. The applied voltage is 50 Volts. Calculate the capacitance, the stored charge, the electric field, the stored energy and the force between the plates.

Problem 2 Consider a capacitor with plate area of 100×100 μm having a gap distance of 5 μm. The capacitor has vacuum between the plates. On one plate there is an oxide (SiO_2) of thickness 0.5 μm and the spring constant is $k = 20$ N/m. Calculate the pull in and the pull out voltage.