CHAPTER

THREE

Flow physics

The theory of fluid dynamics is quite a complicated area. The early theories date back to the seventeenth century, and it is still a field in both theoretical and experimental advance. In the human circulatory system very complicated, non-stationary flow patterns arise, and this chapter can only scratch the surface of this vast topic. It is, however, the hope to introduce the reader to some of the basic concepts. Section 3.1 gives a brief tour of the human circulatory system along with some data. The basics of stationary flow are introduced in Section 3.2, including Bernoulli's equation, viscosity, and Poiseuille flow. The characterization of the transition to disturbed and turbulent flow through Reynolds numbers is given in Section 3.3. Entrance effects at positions of sudden geometric changes in vessel dimensions are introduced in Section 3.4. Sections 3.5 and 3.6 are devoted to pulsatile flow in rigid and elastic tubes, respectively, and show how velocity profiles can be predicted from a measurement of the volumetric flow rate. Further, the effect of an elastic wall is dealt with, leading to the concepts of pulse propagation velocity. Finally, the flow relations at branching and curving vessels are elucidated in Section 3.7.

3.1 Human circulatory system

The human circulatory system is responsible for carrying oxygen and nourishment to the organs and for disposing of the waste products resulting from metabolism. A schematic diagram of the system is shown in Fig. 3.1. The pumping action is carried out by the heart, which contains four chambers: the left and right atria and the left and right ventricles. The blood is ejected from the left ventricle through the aorta and passes through numerous branches of the arterial tree to all parts of the body. Initially, the blood passes through the arteries which branch into smaller vessels called arterioles. They supply a mesh of microscopic vessels, capillaries, where the interchange of nourishment and deposit of waste products between the blood and organs take place. The capillaries assemble into a series of venulae that supply the veins, which carry the blood back to the heart. This is the systemic circulation. The pulmonary circulation carries the blood through the lungs. The blood enters the right atrium and is ejected into the right ventricle and pumped on to the lungs for oxygenation in the lung capillaries. The blood is supplied back to the left atrium by the pulmonary veins.

The arterial and venous trees are shown in Fig. 3.2. The arterial walls are very flexible and contract and expand in response to the pulsation of the blood. The contraction or expansion of the arteriole determines the blood pressure and the pressure in the capillaries, where no pulsation is seen.

The main veins in the body are shown in Fig. 3.2. The veins have thinner and less elastic walls, but also have a larger diameter than the corresponding arteries, and the veins function as a blood reservoir. Most of the veins in the extremities and the neck also have valves to hinder back-flow. Some characteristic vessel dimensions are given in Table 3.1. From the figures and the table, it is seen that the circulatory system consists of vessels of diverse sizes and that curving and branching of vessels takes place through-

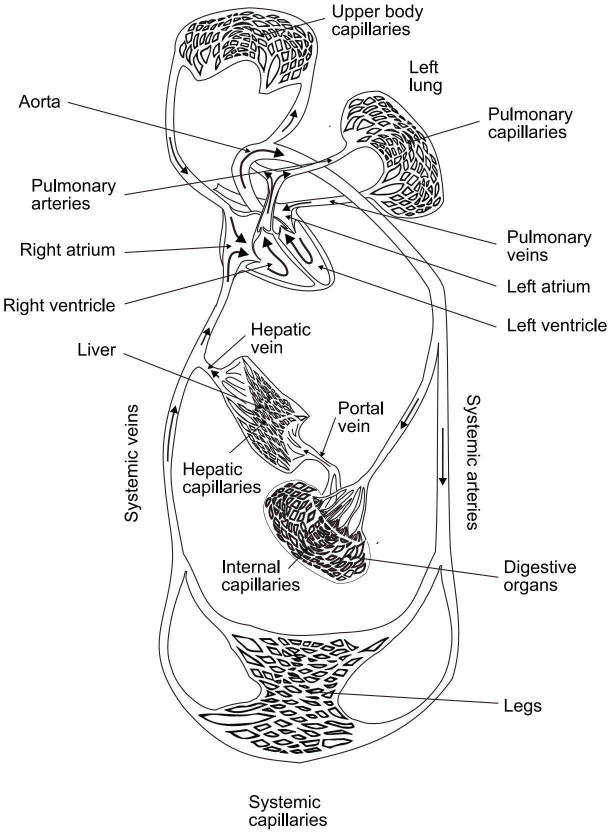

Figure 3.1: Diagram showing the course of the circulation of blood in the human body.

Table 3.1: Typical dimensions and flows of vessels in the human vascular system (data taken from Caro et al. (1974))

Vessel	Internal diameter cm	Wall thickness cm	Length cm	Young's modulus $N/m^2 \cdot 10^5$
Ascending aorta	1.0 – 2.4	0.05 – 0.08	5	3 – 6
Descending aorta	0.8 – 1.8	0.05 – 0.08	20	3 – 6
Abdominal aorta	0.5 – 1.2	0.04 – 0.06	15	9 – 11
Femoral artery	0.2 – 0.8	0.02 – 0.06	10	9 – 12
Carotid artery	0.2 – 0.8	0.02 – 0.04	10 –20	7 – 11
Arteriole	0.001 – 0.008	0.002	0.1 – 0.2	
Capillary	0.0004 – 0.0008	0.0001	0.02 – 0.1	
Inferior vena cava	0.6 – 1.5	0.01 – 0.02	20 – 40	0.4 – 1.0

Vessel	Peak velocity cm/s	Mean velocity cm/s	Reynolds number (peak)	Pulse propagation velocity cm/s
Ascending aorta	20 – 290	10 – 40	4500	400 – 600
Descending aorta	25 – 250	10 – 40	3400	400 – 600
Abdominal aorta	50 – 60	8 – 20	1250	700 – 600
Femoral artery	100 – 120	10 – 15	1000	800 – 1030
Carotid artery				600 – 1100
Arteriole	0.5 – 1.0		0.09	
Capillary	0.02 – 0.17		0.001	
Inferior vena cava	15 – 40		700	100 – 700

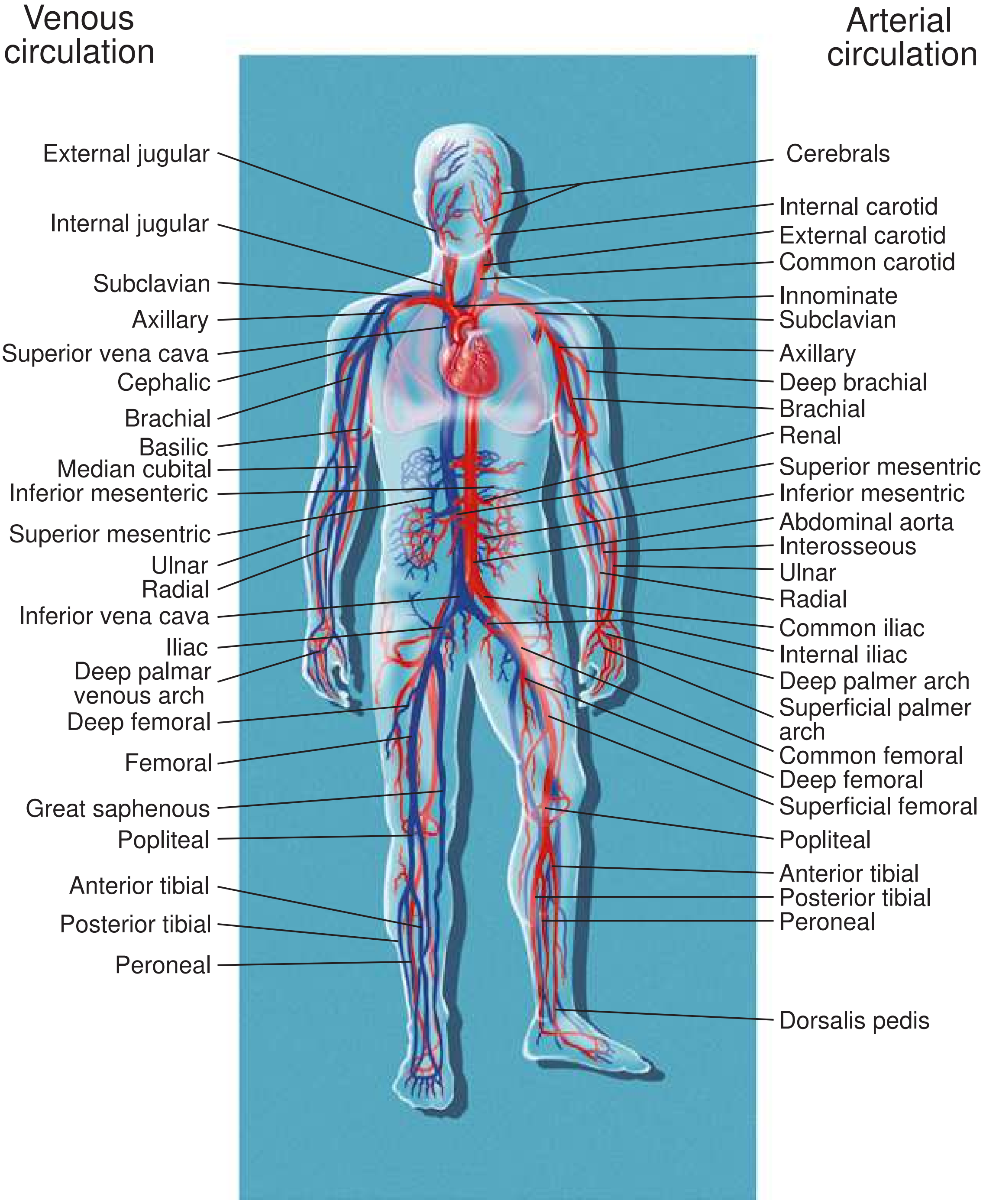

Figure 3.2: Major arteries and veins in the body (reprinted with permission from Abbott Laboratories, Illinois, USA).

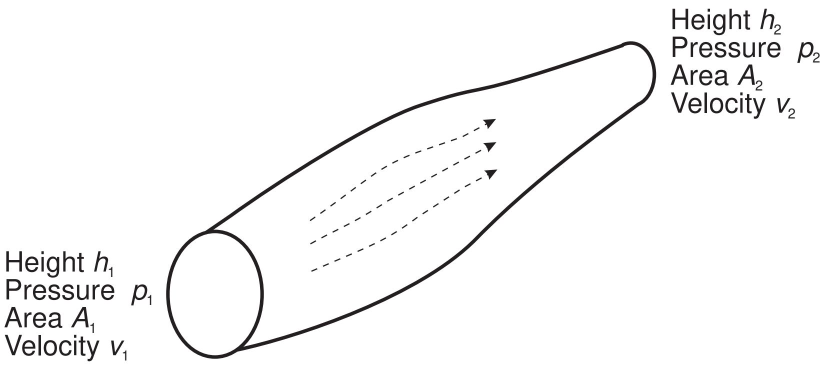

Figure 3.3: Flow in a narrowing tube.

out the vascular system. Further, it must be stressed that the flow is pulsating, so very complex flow patterns are encountered.

The normal pulsatile flow of blood can be disturbed through a number of pathological conditions. A very common condition that evolves with age in humans is the formation of plaque within the vessels. Atherosclerotic plaque is the deposit of lipids in the inner layer of the arterial wall. As the plaque matures it may calcify and become fibrotic. The plaque constricts the vessel lumen and changes the flow conditions. The smaller cross-sectional area increases the blood velocity and can lead to disturbed or turbulent flow.

Another circulatory problem evolves in the heart valves. It is common that they do not close completely and a small back-flow (regurgitation) takes place during the heart cycle. The condition usually causes no harm unless the closing operation is seriously hampered. The impaired closing operation then changes the pressure relations in the heart and diminishes the pumping operation, and thereby the cardiac output.

These two conditions are only a few of many that can change the already complicated pulsatile flow in the body. Both can be diagnosed using ultrasound and the velocity estimation techniques described in this book. A number of other examples, and how to use ultrasound for hemodynamic diagnosis, is given by Hatle and Angelsen (1985).

3.2 Steady flow

The blood flow in the body is quite complex, as indicated in the previous section. It is an arduous task to derive a precise description through the complete Navier–Stokes equations for the blood circulation. A lot can, however, be learned from more simple approaches. Starting from the basics, we will consider steady flow in a rigid tube. Initially, a non-viscous fluid will be assumed, so that the velocity is constant over a cross section of the tube. Further, the velocity is assumed steady, *i.e.,* the velocity, v, at one particular position in the fluid never changes

$$\frac{\partial v}{\partial t} = 0, \tag{3.1}$$

but the velocity can change with position, so ∇v is different from zero at places of change in tube geometry.

An example of a steady flow in a tube is shown in Fig. 3.3. The velocity at height h_1 is v_1 over the cross area A_1, the velocity at h_2 is v_2, and the cross-sectional area is A_2. Since no fluid enters or leaves the tube other than at the ends, the amount of fluid entering at A_1 must leave at A_2. This conservation of mass yields:

$$A_1 v_1 \rho_1 \Delta t = A_2 v_2 \rho_2 \Delta t, \tag{3.2}$$

where $v_1 \Delta t$ is the distance traveled by the fluid in a time interval Δt, and ρ_1 and ρ_2 are the densities at h_1 and h_2. Assuming the fluid to be incompressible gives:

$$A_1 v_1 = A_2 v_2. \tag{3.3}$$

If the velocity varies over the cross sections, which will be the case for a real (viscous) fluid, the equation changes to:

$$A_1 \bar{v}_1 = A_2 \bar{v}_2. \tag{3.4}$$

Here $\bar{v}$ is the spatial average of the velocity over a cross section and is:

$$\bar{v} = \frac{1}{A} \int_x \int_y v(x, y) dy dx. \tag{3.5}$$

The volumetric flow rate, Q, through the cross section is then:

$$Q = A\bar{v} = \int_x \int_y v(x, y) dy dx. \tag{3.6}$$

This is the volume of blood flowing per unit time, and has the SI unit of m^3/s. Usually, ml/min is used in hemodynamics.

3.2.1 Bernoulli's law

A fluid confined to flow in a tube will obey the law of conservation of energy. Thus, the sum of mechanical (kinetic and potential) energy and thermal energy is constant. This is expressed by Bernoulli's equation. It is quite easy to derive by using the conservation of mass law and considering the energy of the flow (Feynman et al. 1964).

The work done on the fluid entering at h_1 is $p_1 A_1 v_1 \Delta t$ and on the fluid leaving at h_2 is $p_2 A_2 v_2 \Delta t$. The difference in work is equal to the increase in energy per unit mass times the mass:

$$p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t = \Delta M (E_2 - E_1). \tag{3.7}$$

The energy is equal to kinetic, potential, and internal energy. The last is in the form of thermal energy (heat).

$$E = (\frac{1}{2} m v^2 + gmh + U') \frac{1}{m} = \frac{1}{2} v^2 + gh + U, \tag{3.8}$$

where U is internal energy per unit mass. Combining (3.7) and (3.8) gives:

$$\frac{p_1 A_1 v_1 \Delta t}{\Delta M} - \frac{p_2 A_2 v_2 \Delta t}{\Delta M} = \frac{1}{2}(v_2^2 - v_1^2) + g(h_2 - h_1) + U_2 - U_1. \tag{3.9}$$

The mass flowing in Δt seconds is $\Delta M = \rho_1 A_1 v_1 \Delta t = \rho_2 A_2 v_2 \Delta t$, so:

$$\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} = \frac{1}{2}(v_2^2 - v_1^2) + g(h_2 - h_1) + U_2 - U_1 \tag{3.10}$$

or

$$\frac{p_1}{\rho_1} + \frac{v_1^2}{2} + gh_1 + U_1 = \frac{p_2}{\rho_2} + \frac{v_2^2}{2} + gh_2 + U_2. \tag{3.11}$$

This state equation for fluids makes it possible to quantify the pressure drops due to narrowing or expansion when the velocities are known, or the velocity ratio when pressures can be measured. In many cases the change in thermal energy is neglected, thus, the fluid temperature is fixed. In doing so, the viscosity

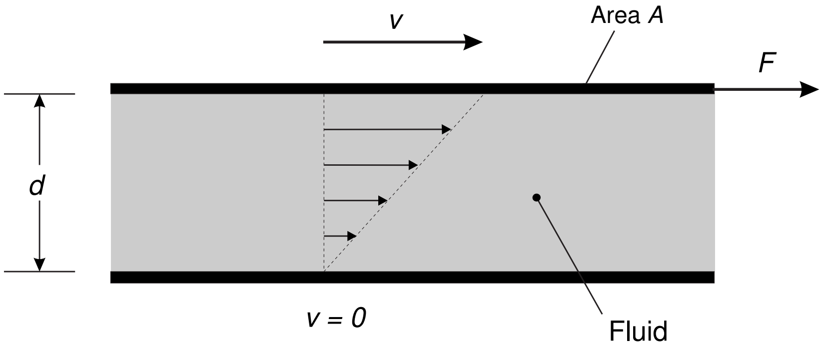

Figure 3.4: Viscous drag between two parallel plates.

(see subsection 3.2.2) of the fluid is discarded. Flow not close to solid boundaries is not much affected by viscosity and, thus, for large vessels the conversion from mechanical to thermal energy is negligible. The loss can be significant and must be accounted for over longer distances and for small vessels.

An example is shown in Fig. 3.3. The conservation of mass gives the relation between the two velocities:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \Rightarrow v_2 = \frac{A_1}{A_2} v_1 \tag{3.12}$$

when the fluid is assumed to be incompressible. Assuming that the internal energy does not change and that $h_1 = h_2$ yields:

$$p_2 = \frac{\rho_2}{\rho_1} p_1 + \rho_2 \frac{1}{2}(v_1^2 - v_2^2) = p_1 + \frac{\rho}{2}\left(1 - \left(\frac{A_1}{A_2}\right)^2\right) v_1^2. \tag{3.13}$$

For $A_1 > A_2$ the pressure will decrease and the velocity will increase.

3.2.2 Viscosity

One of the main properties of any fluid is that it cannot support a static shear stress over time. A shear stress can be applied as shown in Fig. 3.4. The fluid is confined between two plates and a force is applied to the top plate. The fluid will not give a static reaction force to a mere displacement of the top plate, but it will yield a dynamic force in reaction to a constant velocity movement of the plate. This force is in balance with a friction force from shear stress in the fluid, if no other forces are present. The shear stress F/A is proportional to v/d:

$$\frac{F}{A} = \mu \frac{v}{d}. \tag{3.14}$$

The constant μ of proportionately is called the viscosity of the fluid.

In general for a small fluid element, as depicted in Fig. 3.5, the shear stress, which is force per unit area $\Delta F/\Delta A$, is proportional to the rate of change of the shear strain:

$$\frac{\Delta F}{\Delta A} = \mu \frac{\Delta v}{\Delta y} \approx \mu \frac{\partial v}{\partial y}, \tag{3.15}$$

where strain is the ratio between deformation and original length. Accounting for the change in the x-direction also, the general equation is:

$$S_{xy} = \mu \left(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right). \tag{3.16}$$

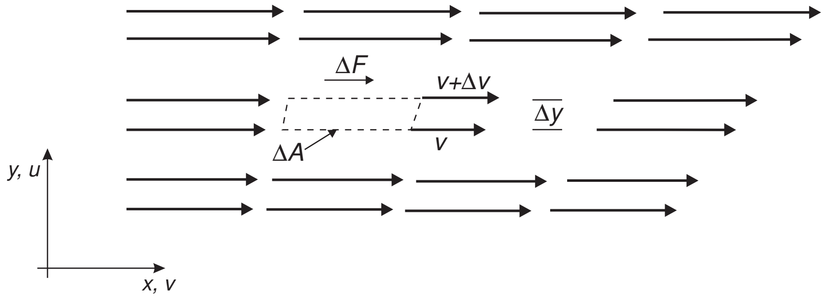

Figure 3.5: Force on small fluid volume in a rigid tube enclosing a steady flow.

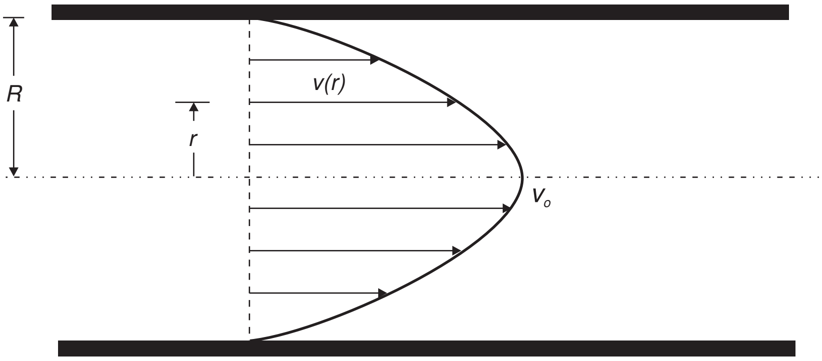

Figure 3.6: Laminar flow in a long, rigid tube.

The SI unit of viscosity is N·s/m^2 or kg/[m·s] although often the unit of centipoise (cP) equal to 0.001 kg/[m·s] is preferred. μ will be independent of the magnitude of S_{xy} for an ideal fluid, which is called a Newtonian fluid. Water is, to a good approximation, a Newtonian fluid, whereas other more complex fluids have non-Newtonian behavior. Blood is a complex fluid consisting of a suspension of cells in plasma, and has a variation in viscosity with shear rate. Variations are also found in small tubes, where blood cannot be characterized as a homogeneous fluid. Both phenomena are fortunately of less importance in this treatment, as significant changes in viscosity are only seen in small vessels (<0.5 mm) and at shear rates lower than that found in the larger vessels in the normal circulatory system (Evans et al. 1989).

The viscosity of blood is dependent on both temperature and composition (hematocrit[1]). An often used value is $4 \cdot 10^{-3}$ kg/[m·s] or 4.0 cP, which is found for normal blood in the larger vessel at a temperature of 37°C and a hematocrit of 45%. This viscosity is four times that of water.

3.2.3 Poiseuille flow

The steady flow of a Newtonian fluid in a long, rigid tube is shown in Fig. 3.6. The flow will be laminar, so that the velocity at a radial position is independent of position along the tube. The distribution of velocities is parabolic and given by

$$v(r) = \left(1 - \frac{r^2}{R^2}\right) v_0, \tag{3.17}$$

where r denotes radial position, R is the radius of the tube, and v_0 is the maximum velocity attained at the center of the tube. Note that the velocity is zero at the walls. This is, in general, true for any kind of velocity profile for a viscous fluid.

[1] The hematocrit is the red blood cells volume to the total blood volume. See also Section 4.4.

The viscosity of the fluid will give a resistance to flow. A pressure difference is, thus, needed to overcome this for maintaining a steady flow. The relation between velocity and pressure difference is given by

$$\Delta P = R_f Q, \tag{3.18}$$

where R_f is the viscous resistance. The relation is named after Poiseuille, who studied flow in capillary tubes and published his findings in 1846 (Nichols and O'Rourke 1990).

For a laminar flow with a parabolic velocity profile the viscous resistance is

$$R_f = \frac{8\mu l}{\pi R^4}, \tag{3.19}$$

where l is the distance over which the pressure drop is found. The resistance is highly dependent on vessel radius. Decreasing the radius by a factor of two increases the resistance by a factor of 16. The resistance given in the equation is only valid for a parabolic profile. Different values are encountered for other profiles.

Poiseuille's law is equivalent to Ohm's law for electrical circuits, and ΔP is equivalent to voltage and Q to current.

Example 3.1 *A straight vessel with a diameter of 2 cm holds a fluid with a viscosity of 4 cP, and rises 20 cm straight up. The mean velocity of the fluid is 30 cm/s, and its density is 1060 kg/m*^{3}.

What is the pressure difference between the ends of the tube?

Initially neglect the viscosity. The pressure difference is then given by Bernoulli's equation

$$\Delta P = \frac{\rho}{2}(v_2^2 - v_1^2) + \rho g(h_2 - h_1).$$

Since the vessel is straight, we have from the conservation of mass that $v_1 = v_2$*, so*

$$\Delta P = \rho g(h_2 - h_1) = 1060 \cdot 9.82 \cdot 0.2 = 2082 \text{ } Pa.$$

The pressure drop due to viscosity is

$$\Delta P = R_f Q = \frac{8\mu l}{\pi R^4}\bar{v}\pi R^2 = \frac{8\mu l}{R^2}\bar{v} = \frac{8 \cdot 0.004 \cdot 0.2}{0.02^2} 0.3 = 4.8 \text{ } Pa.$$

For a vessel with a diameter of 0.2 cm and a mean velocity of 10 cm/s, the pressure drop due to viscosity is

$$\Delta P = \frac{8 \cdot 0.004 \cdot 0.2}{0.002^2} 0.1 = 160 \text{ } Pa.$$

There is, thus, a considerable difference in the pressure relationships between a standing person and a person lying flat on a bed.

3.3 Reynolds number

The flow studied in the last sections was always assumed to be laminar, *i.e.,* the flow proceeds along straight lines in a straight tube. For this flow to be stable, it must be proceeding at a sufficiently slow velocity. Intuitively the viscosity of the fluid and the dimension of the tube must also have an influence on the flow characteristics. A very viscous fluid in a small tube tends to uphold the laminar flow, whereas a very large diameter tube with a low viscosity fluid makes it difficult for the flow to stay laminar (like oil in a small tube and smoke in free air).

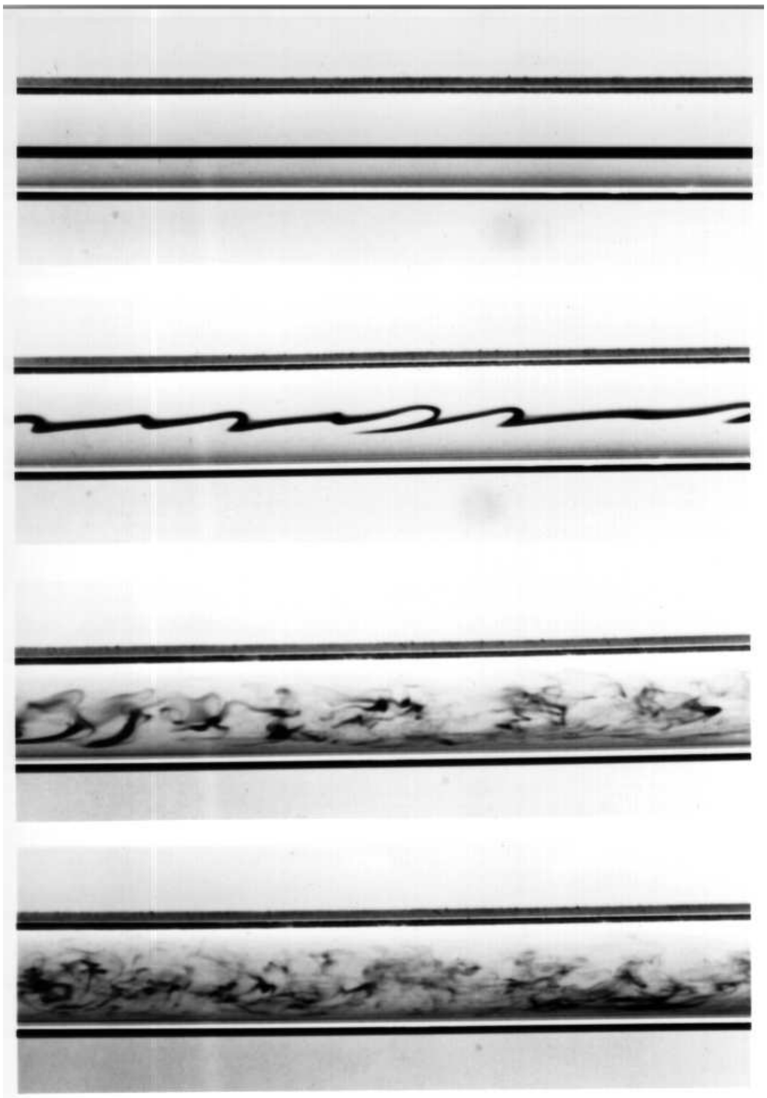

Figure 3.7: Photo of a dye injected into a tube with flowing water. The images are taken of the original apparatus as used by Reynold in 1883 (courtesy of C. Lowe, Manchester School of Engineering).

An indication of whether the flow will be laminar or turbulent is given by the Reynolds number:

$$R_e = \frac{2R\rho}{\mu}\bar{v}. \tag{3.20}$$

Reynolds numbers less than 2000 usually indicate a laminar flow, whereas numbers above 2500 indicate turbulent flow. The number is only a rough indication of how the flow will behave. Laminar flow has been observed at Reynolds numbers up to 50,000 under carefully controlled conditions, and turbulence can be found at $R_e \approx 1000$, if the vessel is not smooth.

An increase in velocity will gradually disturb the flow, so individual particles will not flow along straight lines, but rather start to form small curls and eddies. A further increase in velocity will render the flow turbulent. The stream motion is random and continually changes direction throughout the vessel.

An example of the transition from laminar to turbulent flow is shown in Fig. 3.7. The top photo shows a laminar flow proceeding through the tube in a straight line. Increasing the flow generates slight excursions from the line in the second photo, and a further increase results in turbulent flow, as depicted in the two last photographs.

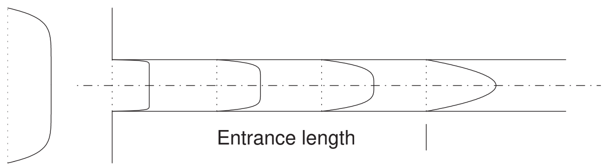

Figure 3.8: Development of a steady-state parabolic velocity profile at the entrance to a tube.

A very interesting observation about the Reynolds number is that different conditions can yield the same number. An increase in velocity can be offset by a shrinkage in tube diameter. It is, thus, possible to make measurements at a smaller scale and then extrapolate to other conditions. This fact is extensively used in, *e.g.*, the aircraft industry.

Example 3.2 *Some typical values for the circulatory system are: Aorta:* $2R$ *= 2 cm,* $\bar{v}$ *= 0.30 m/s, blood:* $\rho = 1.06 \cdot 10^3$ *kg/m*^{3}*,* μ *= 0.004 kg/[m·s], so*

$$R_e = \frac{2R\rho}{\mu}\bar{v} = \frac{2 \cdot 10^{-2} \cdot 1.06 \cdot 10^3}{0.004} \cdot 0.30 = 1590.$$

This Reynolds number indicates that the flow is laminar. Due to the curvature of the aorta and the proximity of the heart valve, the flow is probably disturbed close to the valve, and then gets more laminar away from the heart. For a smaller, peripheral vessel $2R$ *= 0.3 cm,* $\bar{v}$ *= 0.1 m/s, so* R_e *= 159. The Reynolds number tends to drop when going away from the heart, because the total area of the arterial tree expands and the velocity and vessel diameters reduce. The result is a more stable and laminar flow. In general, the Reynolds numbers for the circulatory system are below 2000, and turbulence is an indication of a hemodynamic problem. Especially, a stenotic vessel or clot on vessel walls can increase the velocity and lead to a turbulent or disturbed flow close to the stenosis.*

3.4 Entrance effects

The flow considered in the past sections was assumed to be in a long, rigid tube. Thereby, the parabolic flow profile will have built up due to the viscosity of the fluid. It often happens in the human circulatory system that one vessel branches into two or that a smaller offspring from a larger vessel is found. The profile will not be parabolic at such abrupt changes in geometry. Initially we can assume the profile to be blunt, so all fluid elements have roughly the same velocity. The profile then gradually changes with distance traveled due to the viscosity until it approaches the parabolic shape. An example of this is shown in Fig. 3.8. How long a distance the fluid has to travel before a parabolic profile is established depends on the profile at the inlet. Assuming a flat profile at the start, it has been experimentally determined that the entrance or inlet length is (Nichols and O'Rourke 1990)

$$Z_e = \frac{R \cdot R_e}{15}. \tag{3.21}$$

The various profiles encountered can be approximately described by:

$$v(r) = v_0 \left(1 - \left(\frac{r}{R}\right)^{p_o}\right), \tag{3.22}$$

where p_o is the profile order. For $p_o = 2$ the equation gives the parabolic profile. For increasing p_o the profile gets progressively more blunt, so the velocity is nearly the same over the cross section of

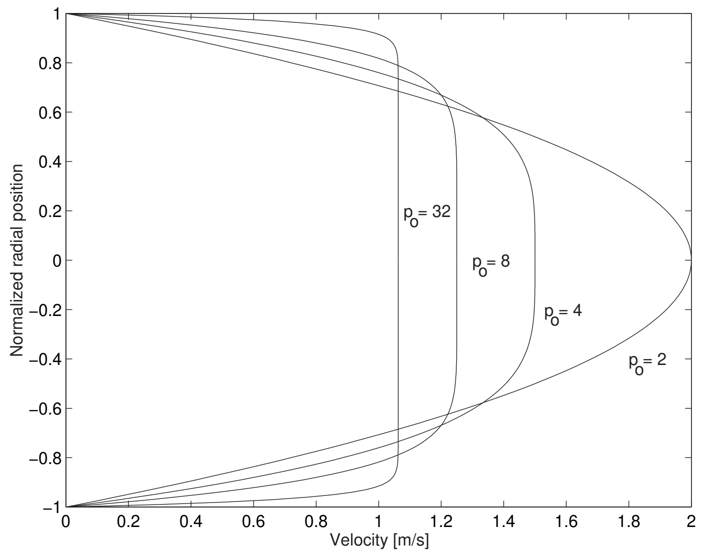

Figure 3.9: Velocity profiles for different values of p_o. The spatial mean velocity is the same for all profiles.

the vessel apart from at the vessel walls, where it is zero. Examples of different profiles are shown in Fig. 3.9.

The spatial mean velocity of the different profiles is:

$$\begin{aligned} \bar{v} &= \frac{1}{\pi R^2} \int_0^R v_0 \left(1 - \left(\frac{r}{R}\right)^{p_o}\right) 2\pi r dr \\ &= \frac{2v_0}{R^2} \left[\frac{r^2}{2} - \frac{1}{R^{p_o}} \frac{1}{p_o + 2} r^{p_o+2}\right]_0^R = v_0 \left(1 - \frac{2}{p_o + 2}\right) \\ &= v_0 \frac{p_o}{p_o + 2}. \end{aligned} \tag{3.23}$$

For a parabolic flow:

$$\bar{v} = \frac{v_0}{2} \tag{3.24}$$

and for a blunt flow

$$\bar{v} = v_0. \tag{3.25}$$

The profiles shown in Fig. 3.9 are normalized to have the same mean velocity. From the conservation of mass, we know that the mean velocity will be the same throughout the tube, if the cross-sectional area is the same and the fluid is assumed to be incompressible. The central core of the fluid will, therefore, accelerate, as the velocity near the walls slows down due to viscosity.

3.5 Pulsatile flow in rigid tubes

The flow in the human body is generated by the pulsatile action of the heart, and a pulsatile rather than steady flow is encountered throughout the circulatory system. The velocity at a point in a vessel is, therefore, dependent on time and experiences acceleration and deceleration. This affects the velocity profile, which cannot be considered parabolic, even when a steady state of pulsation is reached.

Poiseuille's law can be reformulated for pulsatile flow by assuming linearity, *i.e.,* that the flow pattern can be decomposed into sinusoidal components and then added to obtain the velocity variation in time

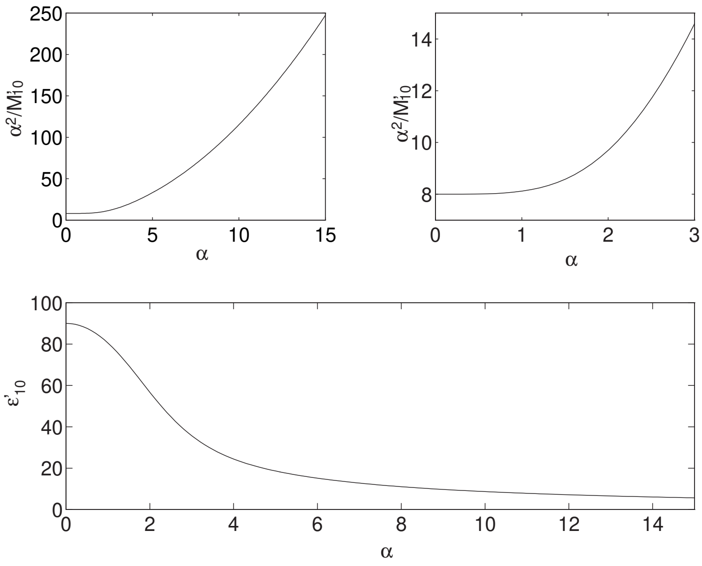

Figure 3.10: The Womersley's parameters α^2/M'_{10} and ϵ'_{10} as a function of α.

and space. This assumes that the fluid is Newtonian. Further, entrance effects are discarded, and it is assumed that the flow has attained a steady state of pulsation. The relation between pressure difference and volume flow rate is then for a single, sinusoidal component (Womersley 1955):

$$\begin{aligned} Q(t) &= \frac{8}{R_f}\frac{M'_{10}}{\alpha^2}\Delta P \sin(\omega t - \phi + \epsilon'_{10}) \\ \alpha &= R\sqrt{\frac{\rho}{\mu}\omega} \\ R_f &= \frac{8\mu L}{\pi R^2} \end{aligned} \tag{3.26}$$

when the pressure difference is $\Delta P \cos(\omega t - \phi)$. Here ϵ'_{10} determines the phase of the wave and M'_{10} its amplitude. They are both complex functions of α. The expressions are given in Appendix A. Womersley's number α indicates how far removed the pulsatile flows pressure-flow rate relation is from that of steady flow. For $\alpha < 1$ the flow rate and pressure is in phase, whereas for $\alpha > 1$ the flow rate will lag the pressure difference.

The evolution of α^2/M'_{10} and ϵ'_{10} as a function α is shown in Fig. 3.10. α^2/M'_{10} becomes equal to 8 for α approaching zero and ϵ'_{10} approaches 90°, so that (3.26) takes the form of the Poiseuille's equation for steady flow. An increase in frequency will increase α^2/M'_{10} and demand a larger pressure difference to obtain the same volume flow rate. The phase difference between pressure and flow rate will also increase and approach 90° for high frequencies.

Assuming a heart rate of 62 beats per minute, α is equal to 10.2 for the first harmonic in the aorta, and 1.69 in a small peripheral vessel (R = 0.5 cm). The pressure difference and volume flow rate will, thus, be nearly 90° out of phase in the aorta and will be in phase in the peripheral vessel. Higher harmonics of the flow rate in the aorta will be severely attenuated due to viscosity, as can be seen in the graph for α^2/M'_{10}. A discussion of the validity of the Womersley equations is given by Nichols and O'Rourke (1990).

From a knowledge of volume flow rate, Evans (1982b) has shown that it is possible to calculate the velocity profile for the steady-state pulsatile flow, when entrance effects are neglected. The relation is

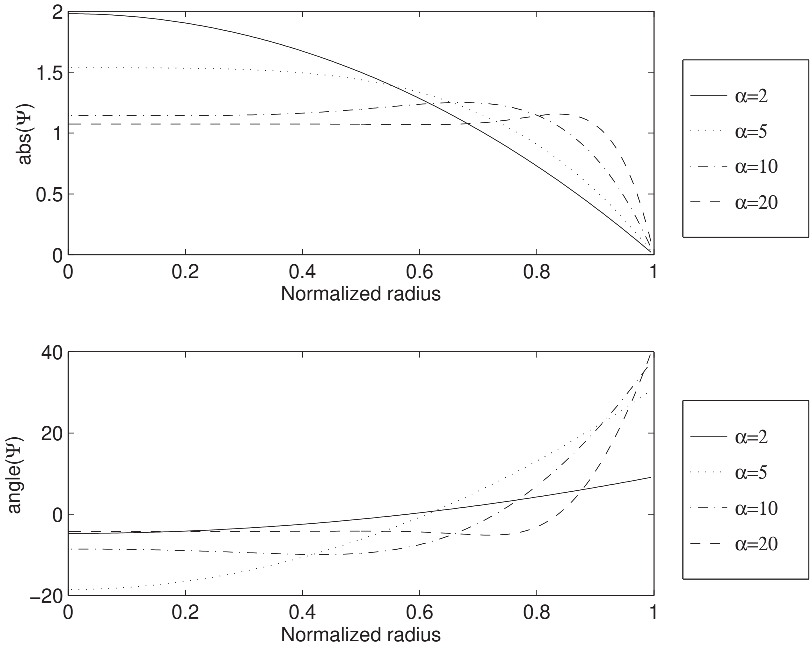

Figure 3.11: Graph of ψ for different values of r/R and α.

given by:

$$\begin{aligned} v_m(t, r/R) &= \frac{1}{\pi R^2} Q_m |\psi_m(r/R, \tau_m)| \cos(\omega_m t - \phi_m + \chi_m) \\ \psi_m(r/R, \tau_m) &= \frac{\tau_m J_0(\tau_m) - \tau_m J_0(\frac{r}{R}\tau_m)}{\tau_m J_0(\tau_m) - 2J_1(\tau_m)} \\ \chi_m &= \angle\psi(r/R, \tau_m) \\ \tau_m &= j^{3/2} R \sqrt{\frac{\rho}{\mu}\omega_m}, \end{aligned} \tag{3.27}$$

where $J_n(x)$ is the nth order Bessel function. Also, $\angle\psi(\frac{r}{R}, \tau_m)$ denotes the angle of the complex function ψ, and $|\psi|$ denotes its amplitude. The function ψ is dependent on radial position in the vessel, angular frequency, and the fluid. It describes how the velocity changes with time and position over a cycle of the sinusoidal flow. Thus, from one measurement of the volume flow rate the whole velocity profile can be calculated.

The evolution of $\angle\psi$ and $|\psi|$ for different radial positions and values of α is shown in Fig. 3.11. Note that the velocity is always zero at the vessel boundary. For α low a nearly parabolic profile is obtained, and the profile becomes progressively more blunt with an increase in α, so the central core of the fluid will move as a whole.

The volume flow rate is related to the mean spatial velocity by:

$$Q(t) = A\bar{v}(t). \tag{3.28}$$

Assuming that the fluid is Newtonian, so linearity can be assumed, it is possible to superimpose the different sinusoidal components to obtain the time evolution of the pulsatile flow. The reverse process is also possible. Here $Q(t)$ or $\bar{v}(t)$ is observed and the individual sinusoidal components found by Fourier decomposition:

$$V_m = \frac{1}{T}\int_0^T \bar{v}(t)\exp(-jm\omega t)dt. \tag{3.29}$$

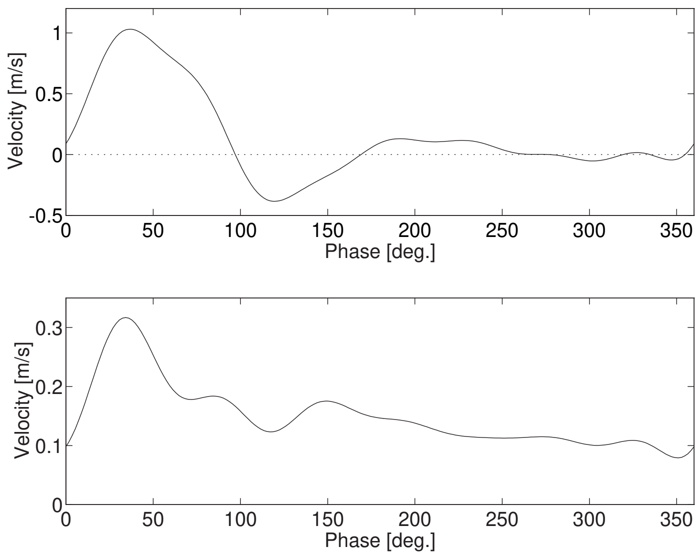

Figure 3.12: Spatial mean velocities from the common femoral (top) and carotid arteries (bottom). The phase indicated is that within a single cardiac cycle.

The spatial average velocity is then given by:

$$\begin{aligned} \bar{v}(t) &= v_0 + \sum_{m=1}^{\infty} |V_m| \cos(m\omega t - \phi_m) \\ \phi_m &= \angle V_m. \end{aligned} \tag{3.30}$$

Using (3.27) and (3.30) it is now possible to reconstruct the time evolution of the velocity profile for the pulsatile flow by:

$$v(t, r/R) = 2v_0 \left(1 - \left(\frac{r}{R}\right)^2\right) + \sum_{m=1}^{\infty} |V_m||\psi_m| \cos(m\omega t - \phi_m + \chi_m). \tag{3.31}$$

The first term is the steady flow.

By observing the averaged flow velocity, the whole profile can be reconstructed. This can be attained by ultrasound systems insonifying the whole vessel and processing the returned signal, as described in Chapters 5 and 6. Examples of the mean velocities from the common femoral and carotid arteries are shown in Fig. 3.12 and the different values for V_p and ϕ_p are given in Table 3.2. The resulting velocity profiles are shown in Fig. 3.13. Profiles for the whole cardiac cycle are shown with time increasing toward the top. The dotted lines indicate zero velocity.

3.5.1 Entrance effects

Entrance effects are also found for pulsatile flow. Again no exact relations can be derived, but the formula (Caro et al. 1978):

$$Z_e \approx 3.4 \frac{v_{core}}{\omega} \tag{3.32}$$

has been found experimentally. Here v_{core} is the core velocity of the flow at the entrance to the vessel, and ω is the angular frequency of the harmonic flow component. Examples of estimated entrance lengths for two vessels are shown in Table 3.3. For the aorta the entrance length for pulsatile flow is much shorter than for steady flow, the entrance length being so long that a fully developed parabolic profile will not be attained. The pulsatile flow can attain the steady state in the aorta, whereas it will not in the renal artery.

Table 3.2: Fourier components for flow velocity in the common femoral and common carotid arteries (data from Evans et al. (1989))

Common femoral					Common carotid				
Diameter		= 8.4 mm			Diameter		= 6.0 mm		
Heart rate		= 62 bpm			Heart rate		= 62 bpm		
Viscosity		= 0.004 kg/[m·s]			Viscosity		= 0.004 kg/[m·s]		
m	f	α	$\|V_m\|/v_0$	ϕ_m	m	f	α	$\|V_m\|/v_0$	ϕ_m
0	-	-	1.00	-	0	-	-	1.00	-
1	1.03	5.5	1.89	32	1	1.03	3.9	0.33	74
2	2.05	7.7	2.49	85	2	2.05	5.5	0.24	79
3	3.08	9.5	1.28	156	3	3.08	6.8	0.24	121
4	4.10	10.9	0.32	193	4	4.10	7.8	0.12	146
5	5.13	12.2	0.27	133	5	5.13	8.7	0.11	147
6	6.15	13.4	0.32	155	6	6.15	9.6	0.13	179
7	7.18	14.5	0.28	195	7	7.18	10.3	0.06	233
8	8.21	15.5	0.01	310	8	8.21	12.4	0.04	218

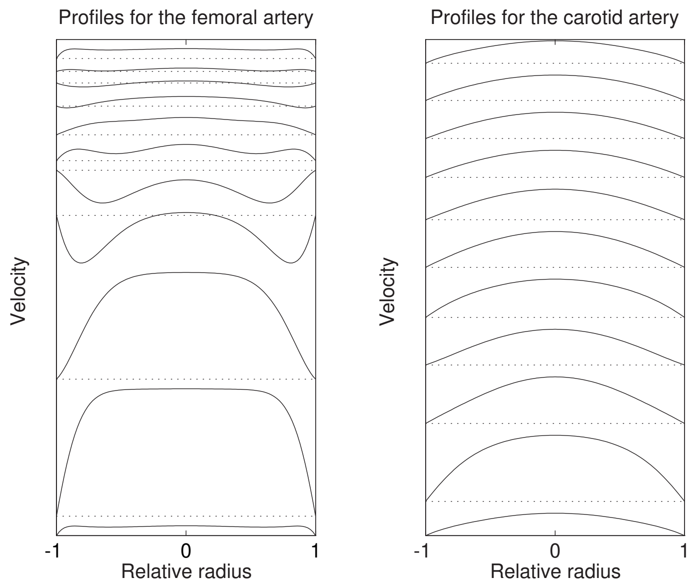

Figure 3.13: Velocity profiles from the common femoral and carotid arteries. The profiles at time zero are shown at the bottom of the figure and time is increased toward the top. One whole cardiac cycle is covered and the dotted lines indicate zero velocity.

Table 3.3: Estimated entrance lengths for the aorta and the renal artery

Vessel	Diameter mm	R_e	$\bar{v}$ cm/s	Frequency Hz	Entrance length cm
Aorta	31	1500	18	DC	140.0
				1	9.7
				2	4.9
Renal artery	6	700	40	DC	13
				1	22
				2	11

3.6 Pulsatile flow in elastic tubes

So far the vessel has been assumed to be rigid, which is not the case for human vessels that are flexible and elastic. The vessels will, thus, expand and contract in response to the pulsatile flow, and this smoothes out the flow. In general, the vessel acts as a transmission line with a low pass characteristic, and higher harmonics of the volume flow rate get progressively more attenuated as they propagate through the vessel.

The elastic tube introduces a propagation speed for the blood pressure wave. If the vessel was rigid and the fluid incompressible, any change in pressure would be immediately propagated throughout the tube. In the human vascular system a pressure change will gradually propagate through the system. An approximate propagation velocity is given by the Moens–Korteweg equation:

$$c_p = \sqrt{\frac{Eh}{2\rho R}}, \tag{3.33}$$

where h is wall thickness, ρ is the density of the wall (roughly $1.06 \cdot 10^3$ kg/m^3), and R is the tube radius. E is Young's modulus for elasticity of the vessel wall in the circumferential direction. The equation was derived under the assumption that the wall is thin, *i.e.,* the ratio $h/2R$ is small. This results in a consistent overestimation of the propagation velocity. A more precise equation is (Nichols and O'Rourke 1990):

$$c_p = \sqrt{\frac{Eh}{2\rho R}(1-\sigma^2)}, \tag{3.34}$$

where σ is the ratio of transverse to longitudinal strain, also called the Poisson ratio. A ratio close to 0.5 is often used, decreasing the propagation velocity by a factor of $\sqrt{3/4}$. Measured velocities and values for Young's modulus are given in Table 3.1. Normally the propagation velocity is 5 to 10 m/s.

3.7 Vessel branching

Vessels in the human circulatory system repeatedly branch and split into a number of smaller vessels. This will change the velocity of the flow and influence the pressures encountered. To assess this situation, consider a steady flow in a rigid tube branching into n smaller tubes of equal dimension, as shown in Fig. 3.14. The velocity is calculated using the conservation of mass. Assuming an incompressible fluid we have

$$\begin{gathered} v_0 \pi R_0^2 = n \pi R_n^2 v_n \\ \Updownarrow \\ v_n = \frac{1}{n}\frac{R_0^2}{R_n^2} v_0. \end{gathered} \tag{3.35}$$

A decrease in velocity takes place if the total area of the small vessels is larger than that for the supply tube. This is, in general, true for the arterial tree in the human body. The cross-sectional area of the tree gets progressively larger and the velocity will diminish as shown in Table 3.1.

The pressure difference needed to overcome the viscous resistance is evaluated by Poiseuille's equation. The flow rates are

$$\frac{\Delta P_0}{R_{f_0}} = Q_0 = nQ_n = n\frac{\Delta P_n}{R_{f_m}}. \tag{3.36}$$

The ratio of the pressure differences per unit length or pressure gradients then is

$$\frac{\Delta P_0/l}{\Delta P_n/l} = \frac{8\mu}{\pi R_0^4} n \frac{\pi R_n^4}{8\mu} = n\frac{R_n^4}{R_0^4} = \frac{nA_n^2}{A_0^2}. \tag{3.37}$$

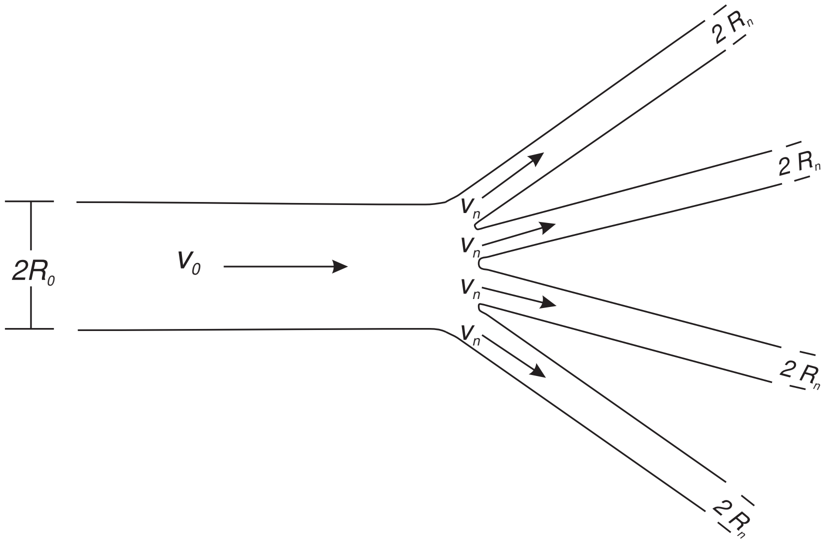

Figure 3.14: Steady flow in a rigid tube supplying n smaller tubes.

The pressure gradient is the same for all tubes if

$$R_n = \frac{R_0}{\sqrt[4]{n}}. \tag{3.38}$$

At a bifurcation

$$R_2 = \frac{R_0}{1.19} \tag{3.39}$$

or

$$A_2 = \frac{A_0}{\sqrt{2}}. \tag{3.40}$$

The area of the vessel, thus, has to increase by a factor of $\sqrt{2}$, if the pressure gradient is to remain the same. The ratio between the total areas of the branching vessels in the human body is on the order of 1.26 (Nichols and O'Rourke 1990), and the increase in pressure gradient for the bifurcation is on the order of $2/1.26^2 = 1.26$. The decrease in velocity is by a factor of 1.26, thus $v_2 = 0.8v_0$.

A bifurcation will also change the velocity profiles encountered. A model of the carotid bifurcation is shown in Fig. 3.15. The initially parabolic profile is split into two, creating profiles skewed toward the inside walls of the vessels. The viscous drag at the walls will then slowly decrease the velocity at the boundaries and start to generate a parabolic profile.

Curving of the vessel will also change the profile. A parabolic profile will be skewed out from the center of the vessel due to centrifugal forces. This also generates a secondary flow pattern, where fluid elements are experiencing a transverse circulation (Caro et al. 1978). A blunt profile undergoes an acceleration of the flow close to the center of curvature to force fluid around the bend. The result is a profile with larger velocities closer to the center of curvature than at the far wall.

Another change in vessel shape found in the human body is that of tapering. Vessels gradually narrow and the Reynolds number is, therefore, successively decreased, and the flow is stabilized.

3.8 Bibliography

A comprehensive treatment of the human circulation is given in Caro et al. (1978) and McDonald (1974). These books give both a theoretical development of the formulae in this chapter as well as comprehen-

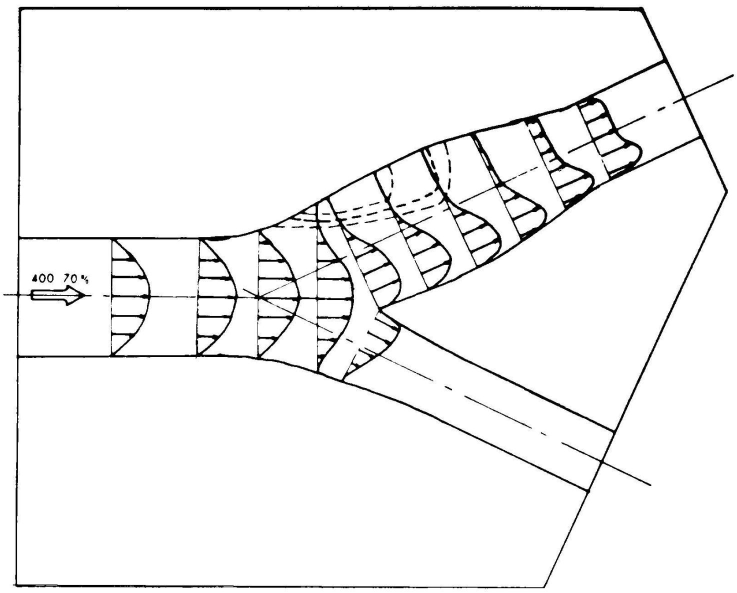

Figure 3.15: Model of the velocity profiles found in the carotid bifurcation. The dashed lines indicate areas with reverse flow (reproduced with permission from Giddens et al. (1985)).

sive data on human circulation and extensive references. The book by McDonald was revised in 1990 by Nichols and O'Rourke after his death. This third version also contains information on the newest measurement techniques and the reference list has been updated.