Blood flow in the human body and its modelling

Flow physics

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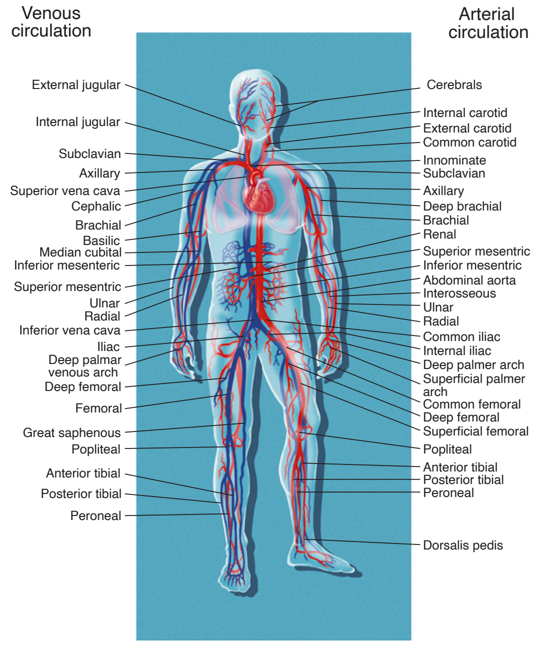
Outline

- **The human cardio-vascular system**
  - The heart
  - The arteries
- **Flow physics**
  - Basic flow characteristics
  - Conservation of mass
  - Navier-Stokes equation
- **Steady flow**
  - The Hagen-Poiseuille equation
  - The Bernoulli equation
- **Pulsatile flow**
  - The Womersley-Evans model



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The circulatory system

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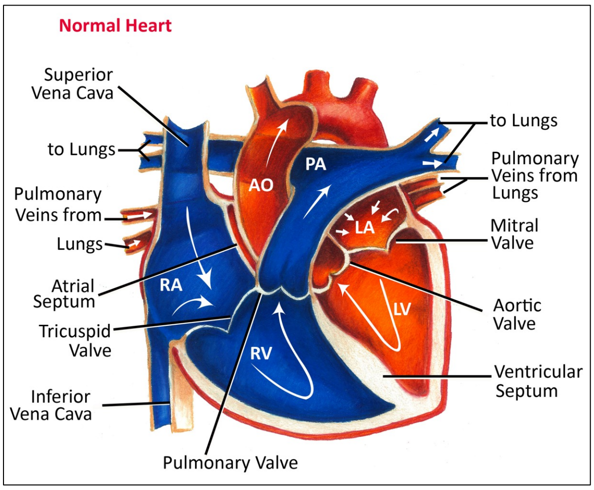
The heart



- Purpose: deliver sufficient amount of blood under sufficient pressure conditions.
- Fine-tuned double pressure pump with high durability and adaptability.
- The heart pumps roughly 6 liters per minute → approximately 9 tons per day.
- LA and RA: volume reservoir + improved inflow through active contraction.
- LV and RV: driving force of the circulation.
- The valves: ensures one-way flow through the heart.

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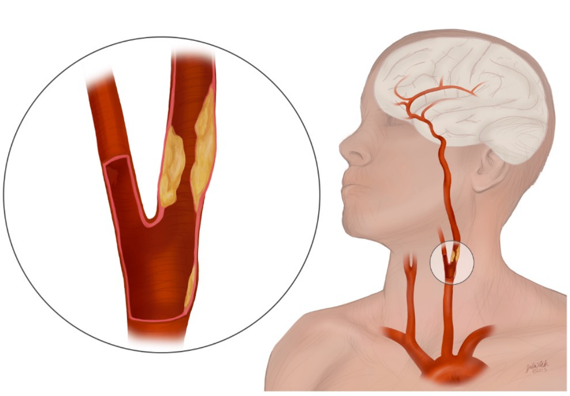
The arteries



- The big arteries act as conduit routes for the flow-based transport of oxygen and carbondioxide.
- The capillaries facilitates the diffusion-based transport of nutrients and waste products.
- Total length of the circulation: 100.000 km
- Total surface area of the capillary bed: 6.000 m^2
- Cardiovascular diseases cover more than 30% of causes of death.

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Basic flow physics

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Definition of fluid mechanics

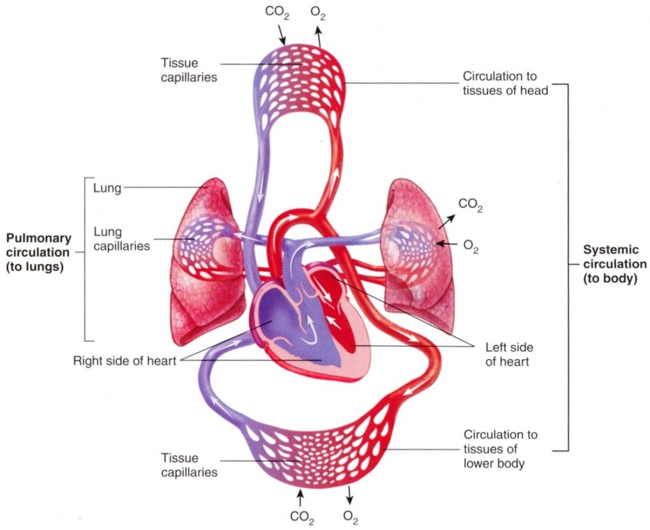
The large blood vessels function as transport routes for the flow-based transport of oxygen and carbon dioxide.

Diffusive substance transport takes place in the capillaries, but this is also dependent on the movement of the blood.

Fluid mechanics (momentum transfer) deals with the **movement of fluids**, such as the blood, and the **forces** that create these movements.

A fluid is defined as a substance that deforms continuously over time when affected by shear stresses.

Fluids are considered a **continuum**, and the purpose is now to define the properties of a fluid element that are central to fluid mechanics.



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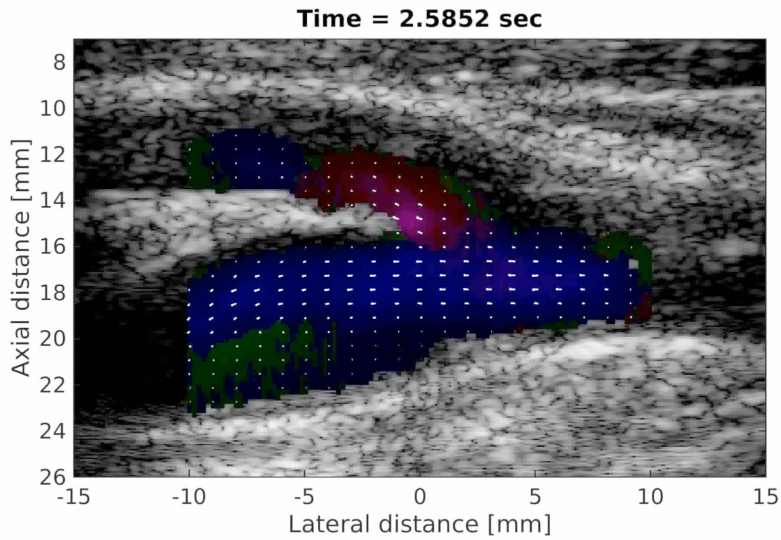
Example: Flow in the carotid bifurcation



The image is created with ultrasound using vector flow imaging (VFI) based on transverse oscillation (TO).

The blood flows from right to left, the colors indicate both the flow direction and the magnitude of the speed (the brighter the color, the higher the speed).

The arrows show the velocity vectors in the flow field.

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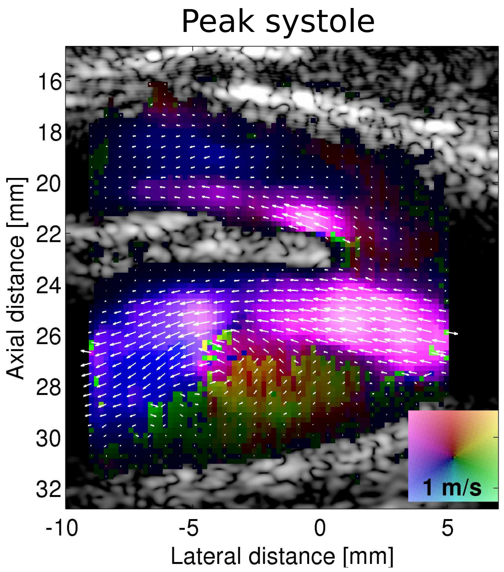
Laminar and turbulent flow

Laminar flow: If particle trajectories in the flow are parallel and do not cross each other, the flow is called laminar.

Turbulent flow: If the particle trajectories cross each other in a chaotic manner, the flow is said to be turbulent.



[1] C.A. Villagomez-Hoyos et al., Accurate angle estimator for high frame rate 2-D vector flow imaging, UFFC, 2016

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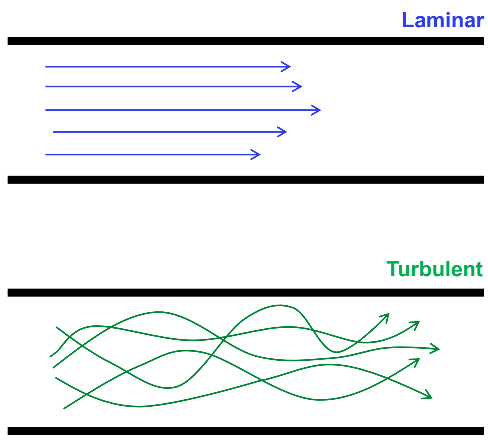
Reynolds number

Reynolds number is used for quantification of the flow type, i.e., laminar or turbulent flow.
It is a dimensionless number which describes the ratio between inertia of the fluid and the viscous forces acting in the fluid.

$$Re = \frac{inertia}{viscous\ forces} = \frac{\rho \bar{v} 2R}{\mu}$$

$Re < 2100$ → Laminar flow

$Re > 4000$ → Turbulent flow



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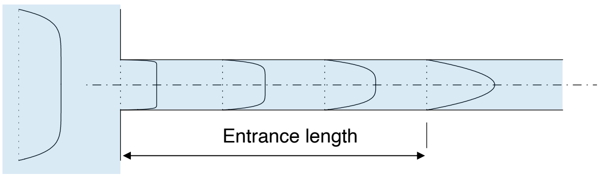
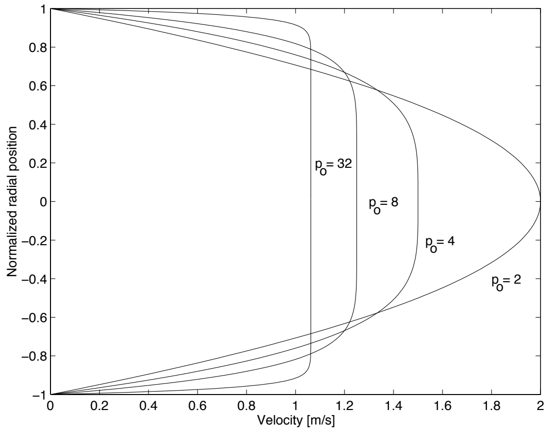
Entrance effects due to no-slip



The entrance length relation has been determined experimentally:

$$L_e = 0.113 \cdot R \cdot Re$$

Where R is the radius of the pipe



The spatial velocity profile can be approximated by:

$$v(r) = v_0 \left(1 - \left(\frac{r}{R}\right)^{p_0}\right)$$

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Entrance effects due to no-slip

The **capillary effect** (or action) plays an important role in how a viscous liquid flows in a pipe.

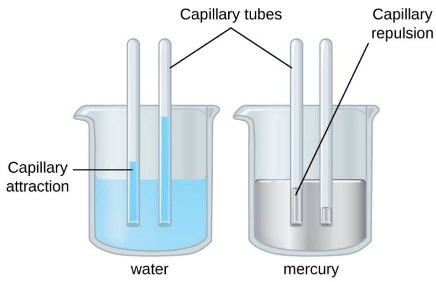
For a wetting fluid, adhesion is greater than cohesion. This means that the outermost thin layer of the liquid sticks to the pipe wall and therefore does not flow. This boundary condition is called **NO-SLIP**.

The consequence of the no-slip boundary condition for flow in tubes is that the velocity of the fluid layer in contact with a solid boundary, i.e., the inner wall of a blood vessel, matches the velocity of the solid boundary:

$$\vec{v}_{fluid} = \vec{v}_{wall}$$

This also means that the velocity of the fluid flowing in a rigid tube will be zero at the tube wall. Due to viscosity effects, a parabolic profile will establish over a given length as long as the cross-section remains constant.

No-slip is also valid in elastic tubes, but compared to rigid tubes the velocity components at the tube wall are not zero as the fluid moves with the wall.



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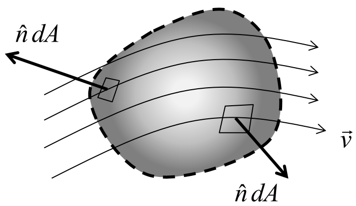
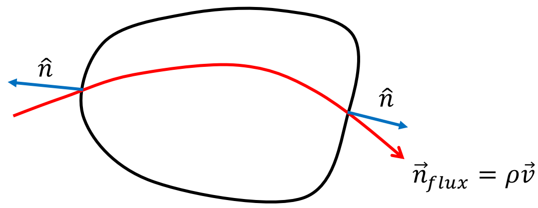
Conservation of mass





We draw up an account of how much mass flows into the control volume and how much mass flows out of the control volume:

$$\begin{Bmatrix} accumulation \\ of\ mass \\ per\ second \end{Bmatrix} = \begin{Bmatrix} mass \\ flux \\ in \end{Bmatrix} - \begin{Bmatrix} mass \\ flux \\ out \end{Bmatrix} + \begin{Bmatrix} net \\ production \end{Bmatrix}$$

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Volume flow rate

Volume flow rate is the volume of liquid that flows through a given surface per unit time. Integration of the mass flux over a surface area gives the volume flow rate.

From conservation of mass:

$$\frac{d}{dt}\int_{V(t)} \rho\, dV = -\int_{S(t)} \rho\vec{v}\cdot\hat{n}\, dA$$

Since blood is incompressible, the density is constant and can be moved outside the integrals.

$$\frac{dV}{dt} = -\int_{S(t)} \vec{v}\cdot\hat{n}\, dA = \dot{Q}_{inlet} - \dot{Q}_{outlet}$$

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Volume flow rate

Since blood behaves as an incompressible fluid under physiological conditions, the volume flow rate in a rigid and closed flow system will be constant.

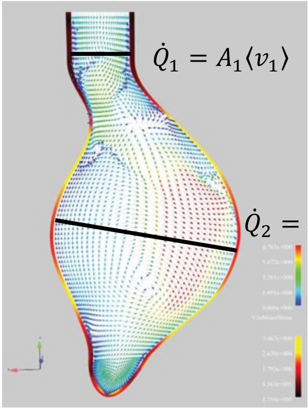
$$\dot{Q}_1 = A_1\langle v_1\rangle = \dot{Q}_2 = A_2\langle v_2\rangle$$

This is used to calculate unknown quantities from measured quantities:

$$\langle v_2\rangle = \langle v_1\rangle\frac{A_1}{A_2} \qquad A_2 = A_1\frac{\langle v_1\rangle}{\langle v_2\rangle}$$

The formulas above are used, for example, to estimate the effective opening area of a stenotic heart valve.

In elastic spaces, like the aneurysm to the right, inflow and outflow are not identical, as the volume of the organ can change. We will need to set up a mass conservation.



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Conservation of linear momentum

Newton's 2nd law (linear momentum balance):

"The time rate of change of momentum of a system is equal to the net force acting on the system and takes place in the direction of the net force"

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Navier-Stokes equation

$$\underbrace{\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right)}_{m\vec{a}} = \underbrace{-\nabla p + \mu\nabla^2\vec{v} + \rho\vec{g}}_{\sum \vec{F}}$$

- Valid for incompressible Newtonian fluids
- Corresponds to Newtons 2^{nd} law of motion applied on fluids
- Relates the velocity field to the pressure drop

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Navier-Stokes equation

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \eta\nabla^2\vec{v} + \rho\vec{g}$$

$\rho\frac{\partial \vec{v}}{\partial t}$ Local acceleration

$+\rho(\vec{v}\cdot\nabla\vec{v})$ Advective acceleration

$=$

$-\nabla p$ Pressure force

$+\eta\nabla^2\vec{v}$ Viscous friction force

$+\rho\vec{g}$ Gravity

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Bernoulli's equation

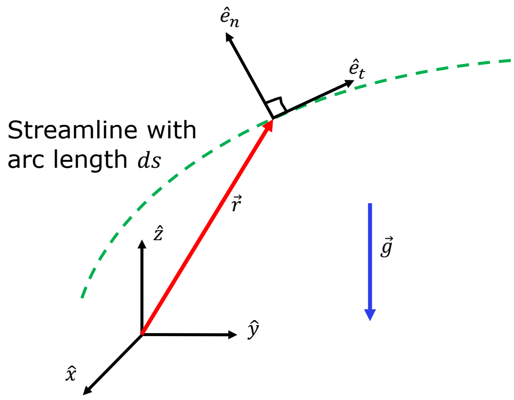
For a frictionless steady flow, the Navier-Stokes equation reduces to

$$\rho\vec{v}\cdot\nabla\vec{v} = -\nabla p + \rho\vec{g}$$

Here it can be seen that the forces arising from advective acceleration of the fluid are balanced by pressure forces and the effect of gravity. Gravity is often assumed to act in the negative z-direction in a Cartesian coordinate system as in the figure to the right. Therefore, gravity can advantageously be expressed as a potential $\phi = zg$ or $\rho\vec{g} = -\rho\nabla\phi$,

$$0 = \rho\vec{v}\cdot\nabla\vec{v} + \nabla p + \rho\nabla\phi$$

Applying this along a streamline yields the component which is tangent to this streamline.



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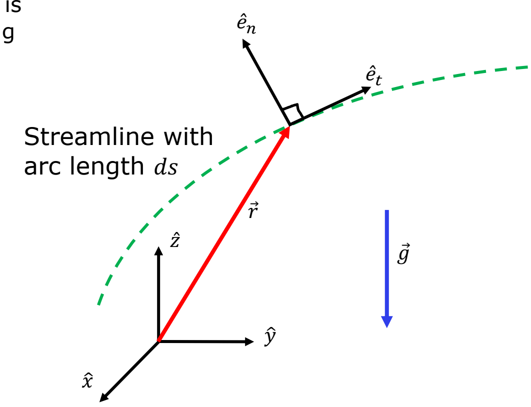
Bernoulli's equation

Finally, the reduced version of Navier-Stokes equation is projected along the streamline and by integrating along the streamline we obtain

$$\frac{\rho}{2}v^2 + p + \rho g z = constant$$



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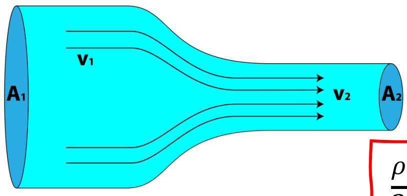
Bernoulli's equation



$$\frac{\rho}{2}v_1^2 + p_1 + \rho g z_1 = constant$$

$$\frac{\rho}{2}v_2^2 + p_2 + \rho g z_2 = constant$$

$$\frac{\rho}{2}v_1^2 + p_1 + \rho g z_1 = \frac{\rho}{2}v_2^2 + p_2 + \rho g z_2$$

$$p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2) + p_2 + \rho g(z_2 - z_1)$$

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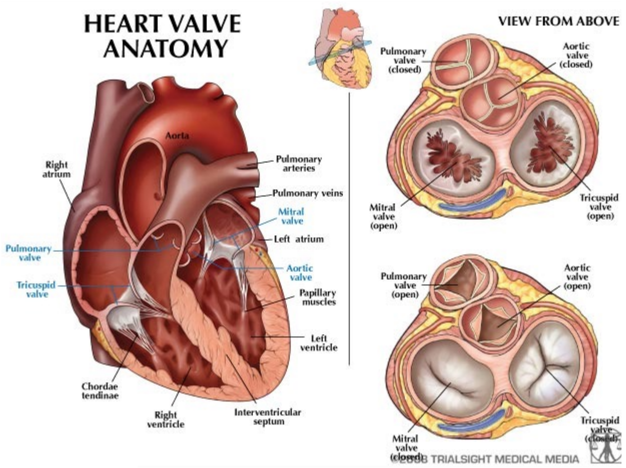
Example: Pressure drop over the mitral valve



Dysfunctional heart valves (stenosis or regurgitation) are relatively common. The most common clinical method for measuring the pressure drop across heart valves is using ultrasound, i.e., echocardiography.

The analysis method is based on Bernoulli's equation. Below is an example of calculating the pressure drop across a heart valve using Bernoulli's equation.

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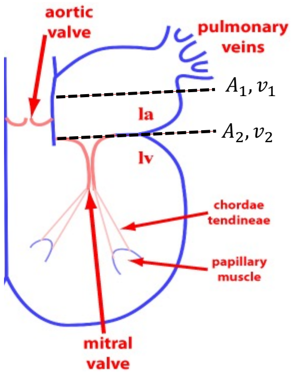
Example: Pressure drop over the mitral valve



$$\frac{\rho v_1^2}{2} + p_1 = \frac{\rho v_2^2}{2} + p_2 \quad \Leftrightarrow \quad p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2)$$

The density of blood is $1060 \frac{kg}{m^3}$ and that corresponds to $8\, mmHg \left(\frac{s^2}{m^2}\right)$. This is used in the equation above to obtain:

$$p_1 - p_2 = 4(v_2^2 - v_1^2)$$

This equation is used daily in hospitals to assess the severity of valvular stenosis or regurgitation.

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Steady flow

The Hagen-Poiseuille model

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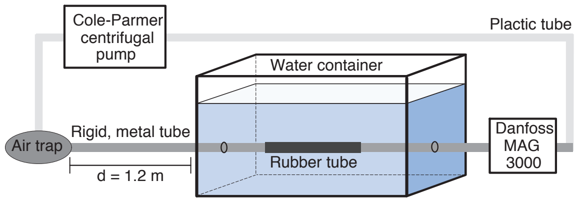
Steady flow



For steady flow:

$$\frac{\partial \vec{v}}{\partial t} = 0 \quad \text{but} \quad \vec{v} \cdot \nabla \vec{v} \neq 0$$

The velocity field does not vary with time but can vary in space due to geometry changes.

Under specific boundary conditions and assumptions, it is possible to obtain an analytical solution to Navier-Stokes equation.

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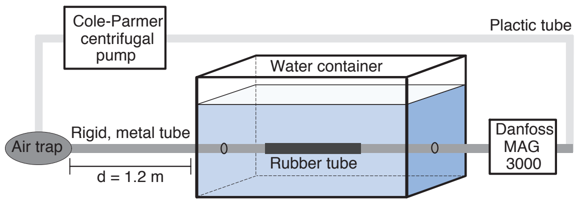
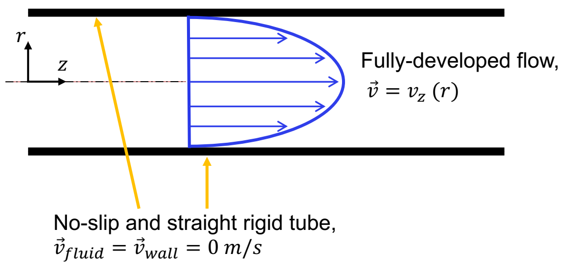
Steady flow





In a long rigid straight tube, the velocity profile of the fully-developed flow can be derived from the Navier-Stokes equation,

$$v_z(r) = \left(1 - \frac{r^2}{R^2}\right) v_0 \quad \text{where} \quad v_0 = v_{max} = \frac{R^2}{4\mu}\frac{dp}{dz}$$

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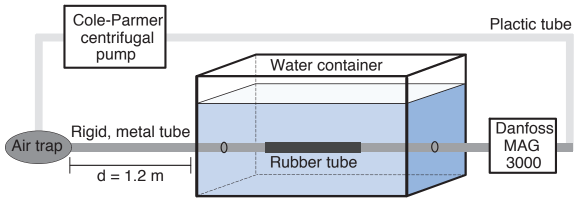
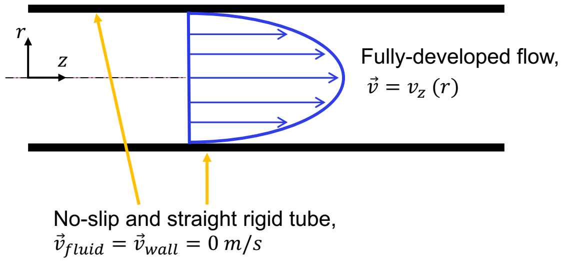
Steady flow





For the steady fully-developed flow there is a unique relation between the peak velocity and the mean velocity:

Peak velocity: $v_0 = \frac{2Q}{\pi R^2}\frac{\Delta p}{L}$

Mean velocity: $\langle v \rangle = \frac{v_0}{2}$

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The Hagen-Poiseuille model

The viscosity of the fluid can be interpreted as a resistance to flow, i.e., the higher the viscosity the more force it requires to make the fluid flow.

It is therefore of interest to quantify the pressure difference needed to overcome this resistance to flow. This can be done for a fully-developed flow in a straight rigid pipe by combining the expression for the peak velocity and the volume flow rate,

$$Q = \frac{v_0}{2}A = \frac{R^2}{8\mu}\frac{\Delta p}{L}(\pi R^2)$$

$$\Updownarrow$$

$$\Delta p = Q\frac{8\mu L}{\pi R^4}$$

This model is known as Poiseuille's law (or the Hagen-Poiseuille model) since the relationship between pressure difference and volume flow rate was determined from experimental measurements performed independently by Hagen and Poiseuille in 1838-40.

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The Hagen-Poiseuille model

Here, we observe that the resistance is highly dependent on the caliper of the vessel, since decreasing the radius with a factor of two, the resistance increases by a factor of 16.

One can draw a parallel between the Hagen-Poiseuille model and Ohm's law for electric circuits where the pressure difference corresponds to the voltage and the volume flow rate corresponds to the current.

$$\Delta p = Q\frac{8\mu}{\pi R^4} \quad \Longrightarrow \quad \Delta p = R_f Q$$

$$R_f = \frac{8\mu l}{\pi R^4}$$

As an example, a 10% decrease in vessel radius will lead to a drop of 46% in volume flow rate under these conditions. Therefore, it is important to monitor changes in the arterial cross-section area because it will influence the resistance downstream and change the pressure conditions.

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Pulsatile flow
The Womersley-Evans model

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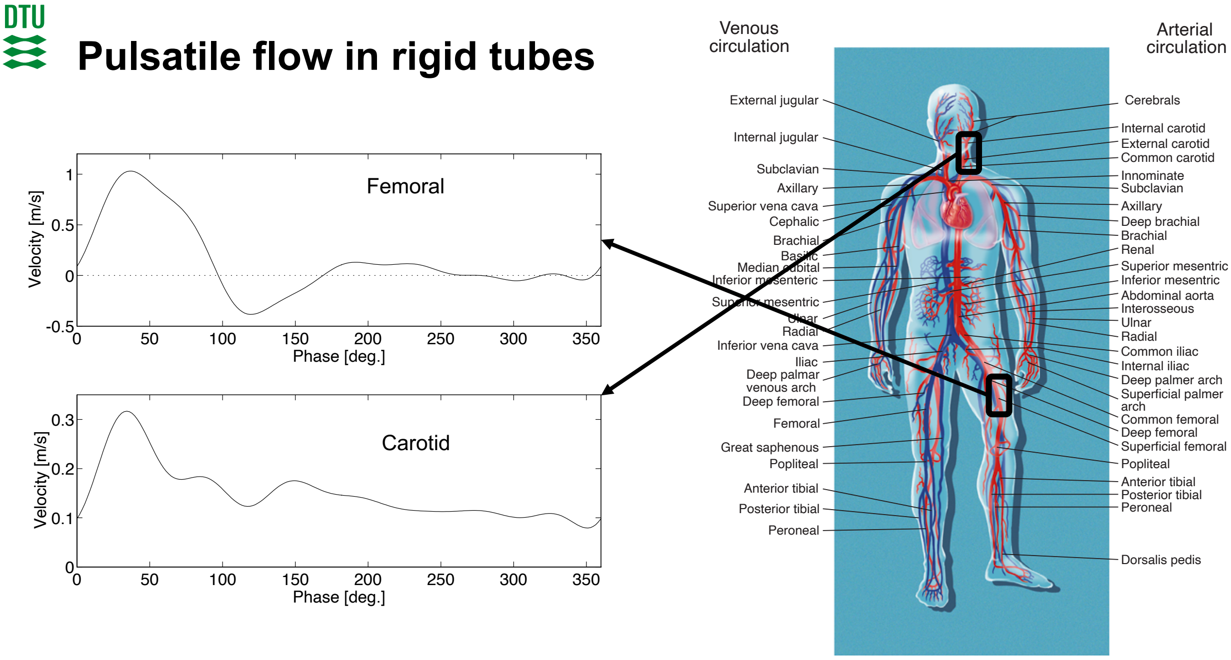
Pulsatile flow in rigid tubes



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Pulsatile flow in rigid tubes

- The velocity field in response to an oscillating pressure field was determined by J.R. Womersley in 1955. Knowledge of the relationship between the pressure and the velocity for a given pressure waveform permits determination of the entire velocity field as a function of both time and space.
- Mathematically, the pressure and velocity waveforms can be decomposed into Fourier series given the following assumptions:
  1) Newtonian fluid
  2) Entrance effects are discarded
  3) The pulsation is steady
- Womersley presented the relation between pressure and velocity for a single sinusoidal component in terms of pressure difference and volume flow rate.

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Womersley's number

Womersley's number is, like Reynold's number, a dimensionless number that describes a certain characteristic of the fluid flow.

Womersley's number is a measure of how far away the pressure-flow rate relation for the pulsatile flow is from that of steady flow, i.e., how large the phase lag is between pressure flow rate.

Womersley's number is given as

$$\alpha = R\sqrt{\frac{\rho}{\mu}\omega}$$

Where ω is the frequency of the oscillating flow

Womersley's relation for a single sinusoidal component of the flow is given as,

amplitude

$$Q(t) = \frac{8}{R_f}\frac{M'_{10}}{\alpha^2}\Delta P \sin(\omega t - \phi + \epsilon'_{10})$$

phase

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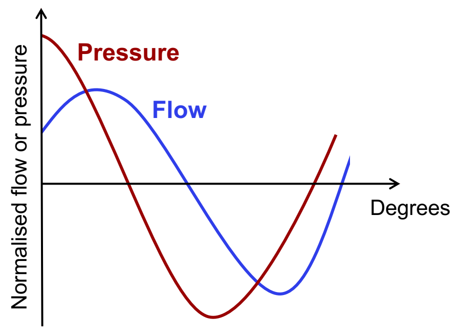
Womersley's number

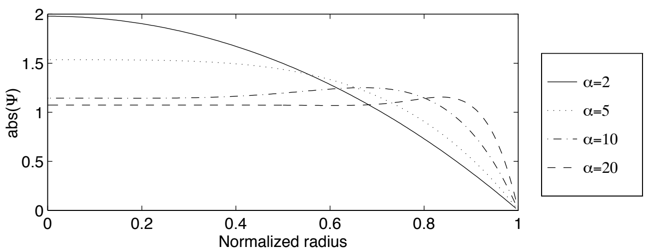
Vessel	Radius	Re	α
Aorta	1.5 cm	1500	21.7
Femoral art.	0.27 cm	180	3.9
Coronary art.	0.425 cm	270	6.15
Arterioles	0.05 cm	17	0.72

From Truskey, Yuan and Katz, "Transport phenomena in biological systems", 2004



When $\alpha < 1$ the pressure and flow rate are in phase, and when $\alpha > 1$ the flow rate will lag the pressure.

As the Womersley number increases the velocity profile of the flow becomes blunter.



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The Womersley-Evans model

In 1982 Evans showed that from knowledge of the volume flow rate it is possible to calculate the velocity profile for a steady pulsating flow when neglecting the entrance effects:

(1) $$v_m(r/R\,,t) = \frac{1}{\pi R^2} Q_m |\psi_m(r/R\,,\tau_m)| \cos(\omega_m t - \phi_m + \chi_m)$$

(2) $$\psi_m(r/R,\tau_m) = \frac{\tau_m J_0(\tau_m) - \tau_m J_0(\frac{r}{R}\tau_m)}{\tau_m J_0(\tau_m) - 2J_1(\tau_m)}$$ where $J_n(x)$ is the n-th order Bessel function

(3) $$\chi_m = \angle\psi(r/R,\tau_m)$$ angle of the complex function ψ

(4) $$\tau_m = j^{3/2} R \sqrt{\frac{\rho}{\mu}\omega_m}$$

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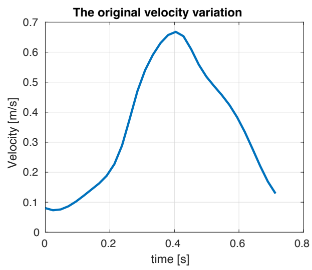
The Womersley-Evans model



Parameter	Value	Description
R	3.5 mm	Vessel radius
ρ	1030 kg/m^3	Fluid density
μ	4.0 mPa·s	Fluid viscosity
T	0.76 s	Length of flow cycle
J	9	Number of harmonics in the spectral decomposition
F	1/T	Harmonic frequency
ω	2πF	Angular frequency
v_0	-	Peak velocity of the profile
r_{sam}	20	No. of radial samples in the reconstruction
t_{sam}	40	No. of temporal samples in the reconstruction

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The Womersley-Evans model

Recall that the volume flow rate is related to the mean spatial velocity profile:

$$Q(t) = Av(t)$$

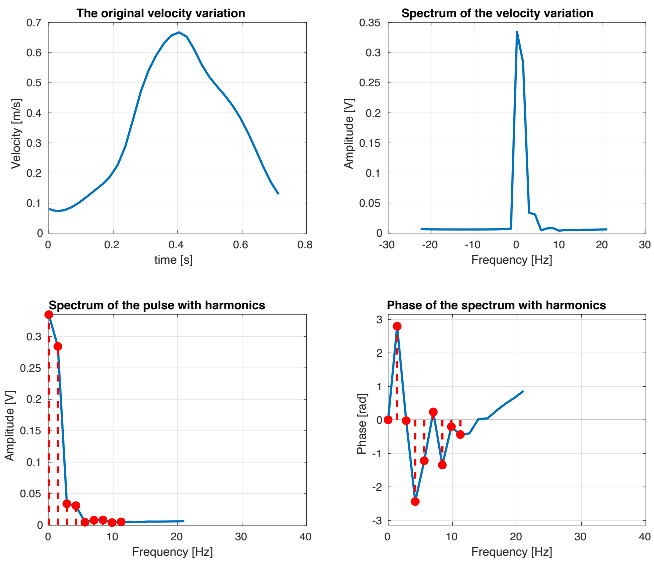
The sinusoidal components are found by Fourier decomposition:

$$V_m = \frac{1}{T}\int_0^T v(t)e^{-jm\omega t}dt \quad (5)$$

The spatial mean velocity is then given by:

$$v(t) = v_0 + \sum_{m=1}^{\infty} |V_m| \cos(m\omega t - \phi_m) \quad (6)$$

$$\phi_m = \angle V_m \quad (7)$$



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Example code

```
%% SETTING PHYSIOLOGICAL PARAMETERS %%
rho = 1.06e3;                  % [kg/m³] density of the blood
mu = 0.0035;                   % [Pa*s] dynamic viscosity of the blood

%% CALCULATION OF PSI %%
% The equations for reconstructing the velocity profiles are given in
% Estimation of Blood Velocities Using Ultrasound: A Signal Processing
% Approach, 2nd Edition on pages 55-58.

% Psi is given in eq. (3.27b)

psi = 0;

for p = 1:J % J is the number of harmonics in the spectral decomposition
    omega = p*omega_0;    % harmonic angular frequency
    tau_p = sqrt(-1)^(3/2)*R*sqrt(omega*rho/mu);
    index = 1;
    % calculating psi for each radial position
    for r_rel = 0:deltaR:0.99
        Be = tau_p*besselj(0,tau_p); % first term in the nominator and denominator
        psi(index,p) = (Be - tau_p*besselj(0,r_rel*tau_p))/(Be - 2*besselj(1,tau_p));
        index = index + 1;
    end
end
```

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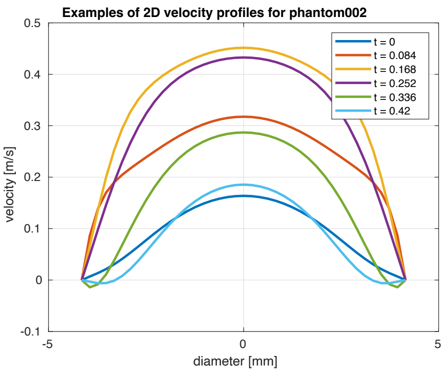
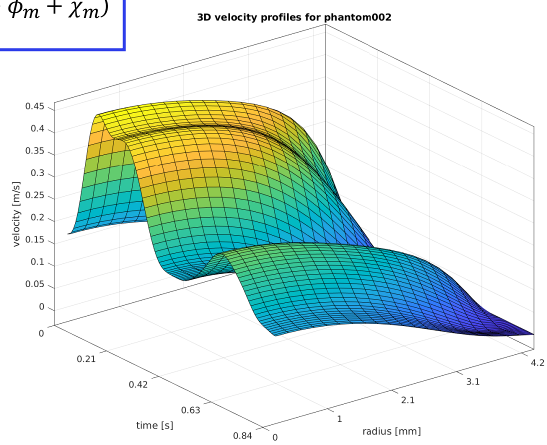
The Womersley-Evans model

Combining equations (1)-(4) with (6)-(7) we get the Womersley-Evans model from which the full velocity field can be reconstructed:

$$v(r/R,t) = 2v_0\left(1-\left(\frac{r}{R}\right)^2\right) + \sum_{m=1}^{\infty} |V_m||\psi_m| \cos(m\omega t - \phi_m + \chi_m)$$





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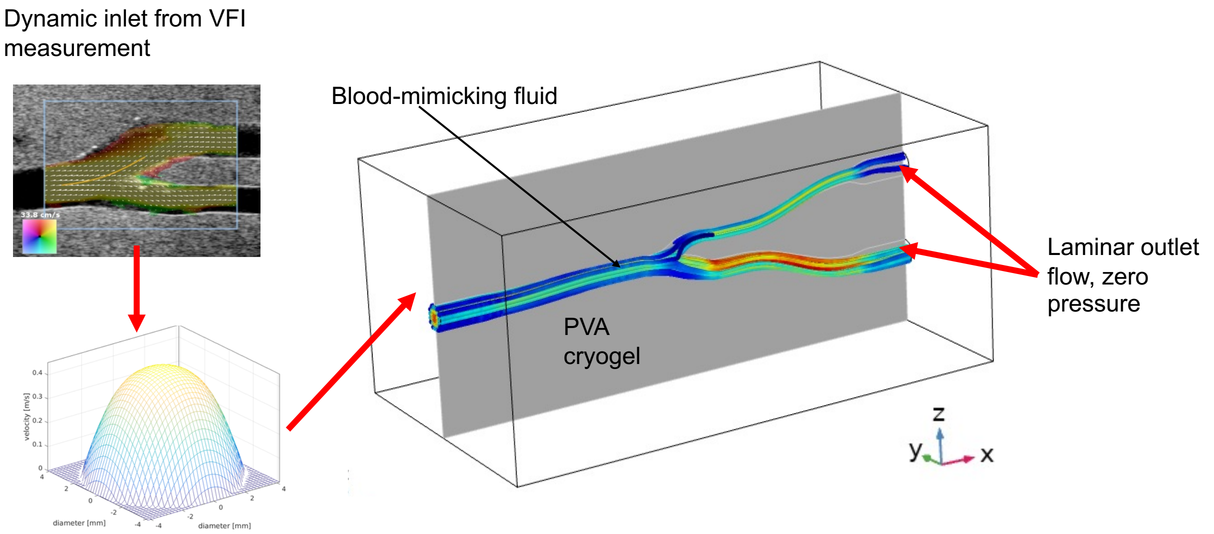
Fluid-structure interaction modelling



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Assignment: Flow physics

- **Purpose:** Use the Womersley-Evans model to reconstruct the full velocity field as a function of time and space.
- **Data:** Simulated RF data mimicking the in vivo flow in the femoral and carotid arteries.
- Builds on top of the exercise of velocity estimation.