BANDWIDTH EXPRESSIONS OF GAUSSIAN WEIGHTED CHIRP

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Indexing terms: Bandwidth, Chirps, Gaussian envelope

This paper calculates four major time duration and bandwidth expressions for a linearly frequency modulated sinusoid with Gaussian shaped envelope. This includes a Gaussian tone pulse. The bandwidth is found to be a non-linear function of nominal time duration and nominal frequency excursion of the chirp signal.

Introduction: When evaluating ultrasound, sonar and radar systems, knowledge of the bandwidth of the transmitted signal is essential. Knowledge of the system bandwidth (transmitted signal and receiver etc.) permits determination of the range resolution and, for Doppler systems, the accuracy in target velocity estimation. In medical pulsed Doppler ultrasound, the envelope of the pulse can often be approximated with a Gaussian shape, and when utilizing frequency modulated (chirp) signals, the envelope of the transmitted chirp can be made Gaussian. It is the aim of this paper to find concise time duration and bandwidth expressions for this important class of signals (*i.e.* linearly frequency modulated sinusoid with a Gaussian shaped envelope).

Definition of chirp signal and spectrum: Diagnostic echo-ranging systems such as radar, sonar and medical ultrasound equipment often transmits either a pure tone pulse or a linearly frequency modulated sinusoidal pulse with a given envelope. In both cases such a signal can be written as

$$g(t) \;=\; g_w(t)\cos[2\pi f_1 t + \pi S_0 t^2] \qquad 0 \le t \le t_m \tag{1}$$

In (1), the first term, $g_w(t)$, is an envelope function of length t_m, while the second is the linearly frequency modulated sinusoid. f_1 is the start frequency and S_0 is the sweep-rate in Hz/s. As the length of the signal is t_m, the instantaneous frequency (of the corresponding analytical signal) at $t = t_m$ is $f_2 = f_1 + S_0 t_m$. The frequency span corresponding to t_m is $\Delta f = f_2 - f_1$. If $S_0 = 0$, (1) defaults to a pure tone. The aim of this paper is to find expressions for the "effective" duration of (1) as well as the bandwidth of the spectrum of (1). To do the latter we introduce the complex version of (1) to obtain a single sided spectrum which next is frequency shifted so as to be placed symmetrically around $f = 0$. $g_w(t)$ must be symmetrical which will be fulfilled by using a Gaussian window, which also permits closed form solutions to the problem. An additional simplification that changes neither the duration nor the bandwidth expressions is introduced by time-shifting (1) so it becomes symmetrical around $t = 0$. The signal in (1) can now be modelled as

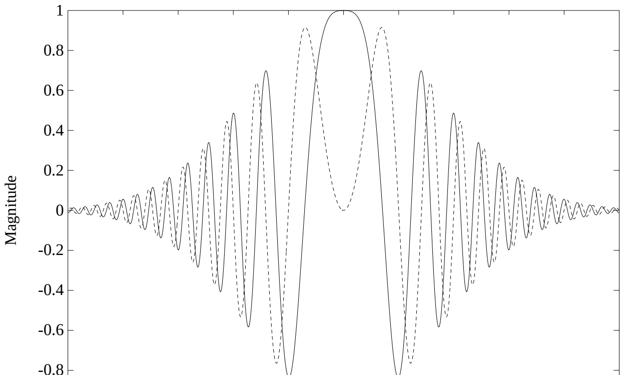

Figure 1. Example of the complex, Gaussian amplitude weighted, linearly frequency modulated sinusoid in (2). Real (—) and imaginary (---) parts are shown. The parameters for this example is: t_m = 100 s, Δf = 1 Hz and α = 3.

$$\tilde{g}_s(t) \;=\; \exp\left[-2\left(\frac{\alpha}{t_m}\right)^2 t^2\right]\,\exp[j\pi S_0 t^2] \;=\; \exp[-(a-jb)\,t^2] \tag{2}$$

where we define $a \equiv 2(\alpha/t_m)^2$ and $b = \pi S_0$. The time duration of $\tilde{g}_s(t)$ is infinite, but the nominal time duration, t_m, is specified such that $|\tilde{g}_s(\pm\, t_m/2)| = \gamma$. For $\gamma \sim 0.0111$, meaning that the amplitude at $t = \pm t_m/2$, has decreased to ~ 1.11 % of the maximal value at $t = 0$, one gets $\alpha = 3$. This α value is also used in [2]. Thus, if the signal length was limited to t_m, $1/\alpha$ would indicate the degree of severity of the truncation. However, in the subsequent formulas, the signal is considered of infinite length. An example of $\tilde{g}_s(t)$ is given in Fig. 1. The spectrum of the signal has been found to be:[1,3]

$$\tilde{G}_s(f) \;=\; \sqrt{\frac{j\pi}{b+ja}}\;\exp\left[-\frac{\pi^2 f^2}{a^2+b^2}(a+jb)\right] \tag{3}$$

Expressions for time duration: Apart from the nominal time duration, t_m, three other expressions for the time duration of the signal can be stated:

ii) The *equivalent rectangular (er)* duration is the duration of a rectangular pulse of the same maximal amplitude and of the same energy as $|\tilde{g}_s(t)|$. It can be expressed as:

$$T_{er} \;=\; \frac{\int_{-\infty}^{\infty} |\tilde{g}_s(t)|^2\,dt}{|\tilde{g}_s^2(0)|} \;=\; \frac{\sqrt{\pi}}{2\alpha}t_m \tag{4}$$

iii) The –3 dB duration is determined by the points where the power has dropped to half of the maximal power:

$$T_{3dB} \;=\; \frac{\sqrt{\ln(2)}}{\alpha}t_m \tag{5}$$

iv) The *root-mean-square (rms)* duration is a measure of the distribution of power around the "point of gravity" of the pulse. It is expressed as:

$$T_{rms} \;=\; \sqrt{\frac{\int_{-\infty}^{\infty} t^2\,|\tilde{g}_s(t)|^2\,dt}{\int_{-\infty}^{\infty} |\tilde{g}_s(t)|^2\,dt}} \;=\; \frac{1}{\sqrt{8}\alpha}t_m \tag{6}$$

These measures are illustrated in Fig. 2. While *ii)* and especially *iii)* are the simplest to calculate from a given signal, *iv)* is the only expression that includes the actual shape of the spectrum. However, as seen from (4) - (6), for a Gaussian shape, the four measures differ only by a constant of proportionality.

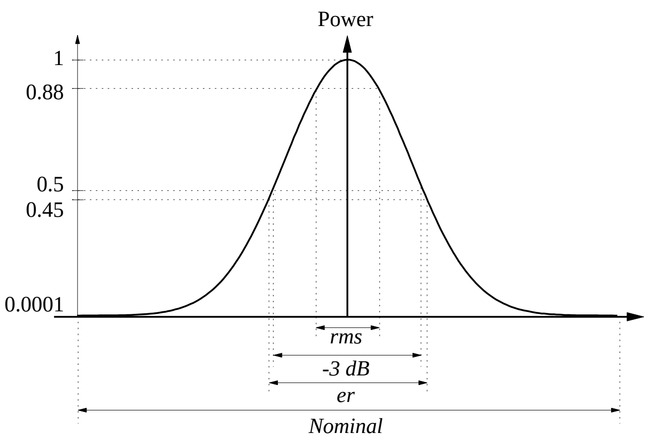

Figure 2. The different width measures indicated on the Gaussian (power) function, $|\tilde{g}_s(t)|^2$ or $|\tilde{G}_s(f)|^2$.

Expressions for bandwidth: Equivalent to the expressions for the time duration, similar expressions exist for the width of the spectrum in (3). These are found by substitution of $\tilde{g}_s(t)$ with

$\tilde{G}_s(f)$ in (4) to (6). The result of this substitution is seen in Table 1. It can be shown that the four bandwidth expressions can be derived from

$$B_x(t_m,\Delta f) \quad = \quad \sqrt{\frac{2\alpha^2}{\pi^2}\left(\frac{1}{t_m}\right)^2 \; + \; \frac{\Delta f^2}{2\alpha^2}} \qquad (7)$$

by use of the coefficients given in Table 1. For example, the *rms* bandwidth is: $B_{rms} = B_x/2$. (7) reveals that the bandwidth expression combine, in a vectorial fashion, contributions from both the finite duration of the signal, t_m, and the frequency excursion, Δf. It can also be shown, that for large time-bandwidth products (slow sweep-rate), $t_m\Delta f$ » $2\alpha^2/\pi$, the envelop of $\tilde{G}_s(f)$ can be approximated by the envelope of $\tilde{g}_s(t)$ properly scaled. This principle can be used for all well-behaving envelopes.[3]

Table 1 Time duration and bandwidth expressions for linearly frequency modulated sinusoid with Gaussian shaped envelope. Also indicated is the minimum time-bandwidth product as well as the energy contained within the given width definition.

Expression	*Nominal*	*er*	–3 dB	*rms*
Duration (s)	$\frac{3}{\alpha}t_m$	$\frac{\sqrt{\pi/4}}{\alpha}t_m$	$\frac{\sqrt{\ln(2)}}{\alpha}t_m$	$\frac{\sqrt{1/8}}{\alpha}t_m$
Bandwidth (Hz)	$\sqrt{18}\,B_x$	$\sqrt{\frac{\pi}{2}}B_x$	$\sqrt{\ln(4)}\,B_x$	$\frac{1}{2}B_x$
min{*TB*} (Δf=0)	$\frac{18}{\pi}$	$\frac{1}{2}$	$\frac{\ln(4)}{\pi}$	$\frac{1}{4\pi}$
Energy, α = 3	>0.9999	0.80	0.76	0.39

With respect to T_{er} and B_{er}, it should be noted that some books[4] give a different, somewhat less tractable, definition of T_{er} so that the minimum time-bandwidth product yields unity instead of a half. Some considerations must also be taken into account when applying the theory to realistic signals. The shape of (3) must be matched as well as possible to the right-hand-side spectrum of the signal in (1). In addition to this, the more the envelope of the time signal deviates from a Gaussian shape (maybe due to truncation), the higher the minimum time-bandwidth product. However, by using Fig. 2, an estimate of time duration and bandwidth can quickly be found.

Conclusions: In this paper expressions for four major time duration and bandwidth expressions have been provided. The numerical differences between these, given a typical (Gaussian) spectral envelope, have been quantified and illustrated. Specifically, the difference between the –3 dB and the *rms* bandwidth is quite large both in nature and magnitude, and it is thus very important that the bandwidth definition is specified together with the value. For a chirp signal with Gaussian envelope, it was found that both the duration of the signal and the frequency excursion contribute to the bandwidth.

Acknowledgements: The author would like to express his appreciation to L. K. Hansen and P. K. Møller for many valuable discussions and S. B. Pedersen and P. C. Pedersen (Worcester Polytechnic Institute, Massachusetts, USA) for valuable discussions, proofreading and comments.

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